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ECONOMETRICA

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A SIMPLIFIED MODEL OF THE CAUSATION OF TECHNOLOGICAL UNEMPLOYMENT

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I. INTRODUCTORY

THE PURPOSE of the present paper is to find the influence on employment of some of the outstanding "data" (extra-economic determining factors). For this purpose, a simplified model has been constructed in which these data and the chief economic variables find their places. Since it is not intended to picture cyclic variations and causations, the model may be called a "long-run model." It excludes some of the most typical cyclical phenomena such as stock-exchange speculation and the existence of small lags of all kinds which are of importance to the explanation of cycles but do not seem to be so for long-run developments.

Since the investigators were interested chiefly in studying the consequences of technological development for employment and the consequences of some of the best-known devices to improve employment, special attention was given to the corresponding sections of economic life.

The calculations have been made for the United States prewar structure (using figures for 1910) and for the postwar pre-Roosevelt structure (using averages for 1919–1932).

II. VARIABLES AND DATA INCLUDED IN THE ANALYSIS

Variables: a = total employment,

b = employment in investment-goods industries,

c = employment in consumers'-goods industries,

u = volume of production of consumers' goods,

$\bar{u}$ = "normal" volume of production of consumers' goods,[1]

v = volume of production of investment goods,

v_{-T} = ditto, T units of time before,

v_e = volume of production of investment goods for expansion of plant,

[1] For explanation, cf. Section III.

p = consumers'-goods price level,
q = investment-goods price level,
$\bar{n}$ = "normal"[1] nonlabor remuneration contained in p,
n' = nonlabor remuneration contained in q,
L = labor income (total wages),
E = nonlabor income,
E' = nonlabor consumption outlay,
E'' = nonlabor savings.

Data:
$\bar{g}$ = "normal" labor quota in unit of consumers' goods,[1]
γ = increase in labor contribution in such unit for increase in production by one unit,
g' = labor quota in unit of investment goods,
h = deflated depreciation allowance per unit of product,
T = lifetime of investment goods,
l = wage rate,
μ = "transition period,"
ΔM = credit creation per time unit,
$\bar{E}$ = "normal" income of nonworkers,
$\bar{E}'$ = "normal" expenditure of nonworkers on consumption,
ϵ = nonworkers' marginal propensity to consume,
t = time.

Some of the terms have to be further explained and so has the choice of the data. This may best be done by discussing, one by one, the relations constituting our "model."

III. THE RELATIONS ASSUMED IN THE MODEL

(1) Starting with total employment a, this may be split up into two parts b and c:

$$a = b + c. \tag{1}$$

(2) Employment b in investment-goods industries will be dependent on the volume v of production of these goods by:

$$b = g'v, \tag{2}$$

where $1/g'$ is labor productivity in this branch. The latter is assumed to be a given quantity, determined by technological development, but independent of v and of the wage rate l. These simplifying hypotheses have not been made for consumers'-goods industries, but since investment-goods industries are far less important it was thought useful not to go into these details here.

(3) For consumers'-goods industries, the relation between employment c and volume of production u is taken as:

$$c = \{\bar{g} + \tfrac{1}{2}\gamma\,(u - \bar{u})\}u. \tag{3}$$

This comes to assuming that the inverted labor productivity or the labor quota per unit of product is a linear function of the volume of production itself; $\bar{u}$ is a reference value of the latter, which may be called the "normal" production. For $u = \bar{u}$, the quantity of labor required per unit equals $\bar{g}$, which is given by technical development. It will be assumed also to depend on wage rates, but since wage rates are also considered as data—for reasons to be set out afterwards—this dependency may be considered later (Section 6, B). There is no serious restriction in the linearity of the function if we consider only small variations in volume of production. The chief reason why inverted productivity depends on the volume of production will be that for a larger production less efficient plants, or parts of plants, or methods, will be necessary.

(4) The volume of investment-goods production v may be split up into two parts, production for replacement and production for expansion of plant; the former is assumed to be equal to total production of investment goods, T time units before, where T is the lifetime of investment goods (including, apart from machines, also buildings and even houses).

We therefore get:

$$v = v_{-T} + v_e. \tag{4}$$

(5) The volume of production of consumers' goods u will be determined, in the first instance, by incomes spent and price level. It is assumed that wages are spent wholly; this leads to:

$$up = L + E'. \tag{5}$$

(6) In a sense this is only a tautology, which has to be completed by an equation telling how consumption outlay E' by nonworkers depends on their incomes; this is assumed to satisfy the relation:

$$E' = \bar{E}' + \epsilon(E - \bar{E}). \tag{6}$$

The relation is linear, which again is no serious restriction if small variations are considered. The coefficients $\bar{E}$, $\bar{E}'$, and ϵ may be called "normal" income, "normal" expenditure, and marginal propensity to consume, respectively.

(7) The two income categories included in the foregoing analysis both depend on economic activity. Total wages L are simply the product of employment a and wage rate l:

$$L = al. \tag{7}$$

Nonlabor income will be discussed later.

(8) The formation of prices may reasonably be treated first. Since long-run relations are considered, prices may be said to equal marginal cost. Marginal cost for consumers' goods will consist of:

(i) Marginal remuneration of nonworkers: $\bar{n}$;

(ii) Marginal labor cost. Since total labor cost equals $cl = \{\bar{g} + \frac{1}{2}\gamma(u-\bar{u})\}\, ul$, marginal labor cost will be found by differentiation with respect to u and be equal to:

$$\{\bar{g}+\gamma(u-\tfrac{1}{2}\bar{u})\}l;$$

(iii) Depreciation allowances. For simplicity and since they are only a small proportion of total costs these are assumed to be independent of the volume of production, but proportional only to prices of investment goods; they will be indicated by hq.

Adding up, we get:

$$p = \bar{n} + \{\bar{g} + \gamma(u - \tfrac{1}{2}\bar{u})\}l + hq. \tag{8}$$

(9) Since investment goods play a less important role than consumers' goods, their prices q are not considered in so much detail. Depreciation allowances will for the prewar case be neglected, and, as before, g' is considered to be independent of v.

$$q = g'l + n'. \tag{9'}$$

For the postwar case, calculations including depreciation allowances have also been made, using the formula:

$$q = g'l + n' + qh. \tag{9}$$

For simplicity, h has been taken equal for both groups of industries. The resulting errors are small.

(10) We are now able to calculate nonlabor income, by subtracting, from the total value of production $pu+qv$, depreciation allowances huq (for postwar calculations $huq+hvq$), and wages

$$\begin{aligned} al &= [\{\bar{g} + \tfrac{1}{2}\gamma(u - \bar{u})\}u + g'v]l; \text{ the result being:} \\ E &= u\bar{n} + \tfrac{1}{2}u^2 l + vn'. \end{aligned} \tag{10}$$

(11) One self-evident relation may be added here:

$$E' + E'' = E. \tag{11}$$

(12) Turning to the sphere of capital formation we have to ask: What funds are available and how are they spent? The funds are:

(i) the stream of savings, E'' per time unit;

(ii) newly created credits, M (provisional notation) per time unit;

(iii) a money stream coming into existence since the accumulated

depreciation allowances are not fully used for replacement—replacement being equal to investment-goods production T time units ago and depreciation allowances being based on plant existing at this moment. These funds yield a stream of $(uh+vh-v_{-T})q$ [in the prewar case $(uh-v_{-T})q$].

This total is spent for two purposes:

(i) the purchase of durable capital goods, as far as they represent new investment; their value is $v_e q$; and

(ii) wage and other income payments connected with an expansion of business. These sums have to be paid for a transition period only; after this period the receipts from increased production will enable the entrepreneurs to pay increased incomes. Indicating the rate of increase in total wages by $\dot{L}$, that in other incomes by $\dot{E}$, and the length of the transition period by μ, the necessary amount—sometimes referred to as increase in circulating capital—will be $\mu(\dot{L}+\dot{E})$. The period μ may be estimated roughly by putting it equal to the circulation period of income, i.e., total money in circulation M, divided by income $L+E$.

The foregoing leads to the equation:

$$E'' + \dot{M} + (uh + vh - v_{-T})q = v_e q + \mu(\dot{L} + \dot{E}),$$

which, since $v = v_{-T} + v_e$, may be written:

$$E'' + \dot{M} + (uh + vh - v)q = \mu(\dot{L} + \dot{E}). \tag{12}$$

In some calculations this will be simplified into:

$$E'' + \dot{M} + (uh - v)q = \mu(\dot{L} + \dot{E}). \tag{12'}$$

Of course, this equation does not tell anything about the motives of investment activity. This question will be considered later (Section IV).

(13) There remains one further equation to be discussed, viz., the one between accumulated investment (in the physical sense) and the normal capacity to produce. Since v represents the volume of production of investment goods per unit of time, and T the lifetime of these goods, there will, at any moment t, be in existence a quantity of $\int_{t-T}^{t} v_\tau \, d\tau$ of them. Per unit of time, $1/T$ of this quantity will normally be "consumed" in the production process. This consumption represents the "contribution," $h(\bar{u}+\bar{v})$, to the normal production $\bar{u}$ and $\bar{v}$ of consumers' and investment goods, respectively, h being the factor introduced as "deflated depreciation allowance."

Thus we get:

$$h(\bar{u} + \bar{v})T = \int_{t-T}^{t} v_\tau d\tau.$$

Since we will find T to be equal to about $2\frac{1}{2}$ units of 10 years, we may replace the integral by:

$$\int_{t-1}^{t} v_\tau d\tau + \int_{t-2}^{t-1} v_\tau d\tau + \int_{t-2\frac{1}{2}}^{t-2} v_\tau d\tau.$$

and since v_t will, in what follows, indicate the volume of production of investment goods in a finite period of which t is the centre, this sum equals, approximately:

$$\tfrac{1}{2}(v_t + v_{t-1}) + \tfrac{1}{2}(v_{t-1} + v_{t-2}) + \tfrac{1}{4}(v_{t-2} + v_{t-3}).$$

Our equation becomes, therefore:

$$h(\bar{u} + \bar{v})T = \tfrac{1}{2}v_t + v_{t-1} + \tfrac{3}{4}v_{t-2} + \tfrac{1}{4}v_{t-3}. \tag{13}$$

Since, in our prewar calculation and in one of the postwar calculations, we neglect depreciation allowances in investment-goods industries, in these cases equation (13) will be replaced by:

$$h\bar{u}T = \tfrac{1}{2}v_t + v_{t-1} + \tfrac{3}{4}v_{t-2} + \tfrac{1}{4}v_{t-3}. \tag{13'}$$

IV. DATA, UNKNOWNS, AND CONSTANTS

The equations (1) to (13) will be used for the description of long-term movements of employment and other economic phenomena. In this description some of the phenomena introduced will be considered as data, others as "phenomena to be explained." Our distinction will not quite coincide with that of usual theory. A few words may therefore be added to defend our choice. We shall consider as data:

(i) The technical coefficients $\bar{g}$, g', h, T, and γ, determining, to some extent, the production function of our model society;

(ii) The psychological coefficients $\bar{E}'$ and ϵ, determining the behavior of consumers (nonworkers);

(iii) The institutional coefficient μ, intimately connected with the velocity of circulation of money;

(iv) The wage rate l and the rate of increase in circulation $\dot{M}$. These will, in general, be considered as economic phenomena, to be explained by theory. Both of them are, however, in present circumstances, highly subject to policy. Our procedure will be to consider them as independent variables and to find out how the choice of their magnitude influences employment and other economic phenomena. If then a certain change in l, say $\Delta' l$, is found to be the most favorable change for a given purpose, it may quite well be that "natural developments," i.e., normal economic forces, lead already to a change $\Delta'' l$; the task of policy then being to complement this $\Delta'' l$ until the total value $\Delta' l$ is reached. For the solution of such problems it is not necessary to know

the "natural development" $\Delta''l$. Similar remarks may be made with respect to $\dot{M}$. The "regular" motives to investment are no longer of importance to its determination as soon as complementary government investments (public works and deficit financing) are included as a possibility.

(v) The remunerations per unit of product $\bar{n}$ of the marginal nonworker in consumers'-goods industries and n' of nonworkers in investment-goods industries. These are considered as constants—and therefore as given—since:

(a) In most literature about technological unemployment not much attention is given to their movements;

(b) Many elements in them will in fact be very sticky, such as interest, rent, and "the adequate income" of an independent entrepreneur. Interest and rent are often fixed for very long times and "the adequate income" is something largely determined by tradition and past experience.

In order to be quite sure that the hypothesis of constant $\bar{n}$ and n' is not a dangerous one, additional calculations have been made where, instead of $\bar{n}$, total nonworkers, income was considered constant—meaning that the remuneration per unit of product varies inversely with production volume u—and the deviations with our case appeared to be small.

Not all data have been supposed to be constant. Apart from $\bar{n}$ and n', just mentioned, this has been assumed to be so for γ, T, $\bar{E}'$, ϵ, and μ. On the other hand, l, $\bar{g}$, g', h, and $\dot{M}$ have been considered as (independently) variable. And the problem solved is that of the change in employment, prices, incomes, and production occurring as a consequence of given changes in the independent variables.

In order to solve this problem it is convenient to combine some of the equations and to differentiate them with respect to time. This latter device is not carried out for equation (12), which already contains differential coefficients. In order to simplify concrete calculations, finite (but small) rates of increase are substituted for differential coefficients. These do not relate, strictly speaking, to time period t, but to the moment between t and $t+1$. For uniformity, equation (12) is also brought into the form relating to that moment, which comes to adding to any term like E'' a term $\frac{1}{2}\Delta E''$. All these operations combined lead us to the following system of equations:

$$\Delta a = \{\bar{g} + \tfrac{1}{2}\gamma(u - \bar{u})\}\Delta u + u\{\Delta\bar{g} + \tfrac{1}{2}\gamma(\Delta u - \Delta\bar{u})\} + v\Delta g' + g'\Delta v, \tag{14}$$

$$\Delta v = \Delta v_{-T} + \Delta v_{e}, \tag{15}$$

$$\Delta p = \{\bar{g} + \gamma(u - \tfrac{1}{2}\bar{u})\}\Delta l + l\{\Delta\bar{g} + \gamma(\Delta u - \tfrac{1}{2}\Delta\bar{u})\} + h\,\Delta q + q\Delta h, \tag{16}$$

$$(1 - h)\Delta q = g'\Delta l + l\Delta g' + q\Delta h, \tag{17}$$

$$l\Delta a + a\Delta l + \epsilon\Delta E = u\Delta p + p\Delta u, \tag{18}$$

$$\begin{aligned} E'' + \tfrac{1}{2}(1 - \epsilon)\Delta E + (hu + hv - v)q + \tfrac{1}{2}(hu + hv - v)\Delta q \\ + \tfrac{1}{2}q(h\Delta u + h\Delta v - \Delta v) + \tfrac{1}{2}qu\Delta h + \tfrac{1}{2}qv\Delta h + \Delta M = \mu(\Delta E + a\Delta l + l\Delta a), \end{aligned} \tag{19}$$

$$\Delta E = \bar{n}\Delta u + n'\Delta v + \gamma lu\Delta u + \tfrac{1}{2}\gamma u^2\Delta l, \tag{20}$$

$$\begin{aligned} \frac{\bar{u} + \bar{v}}{\bar{v}}\Delta\bar{u} = {} & \frac{\frac{1}{2}\Delta v_{-3} + 1\frac{1}{2}\Delta v_{-2} + 2\Delta v_{-1} + \Delta v}{2hT} \\ & - \frac{\frac{1}{2}v_{-3} + 1\frac{1}{2}v_{-2} + 2v_{-1} + v}{2h^2T}\Delta h. \end{aligned} \tag{21}$$

In the establishment of the last of these equations a further hypothesis has been made, in order to avoid consideration of some unimportant details: it has been assumed that $\Delta\bar{u}$ and $\Delta\bar{v}$ show the same proportion as $\bar{u}$ and $\bar{v}$.

As has already been stated in the discussion of the separate equations, several cases have been considered. The above system of equations relates to one (the most complete) postwar calculation. A simpler calculation has been made with postwar and with prewar figures. In these latter calculations depreciation allowances for investment-goods industries have been neglected; this leads to the equations indicated with a prime and gives the following equations instead of (17), (19), and (21):

$$\Delta q = g'\Delta l + l\Delta g', \tag{17'}$$

$$\begin{aligned} E'' + \tfrac{1}{2}(1 - \epsilon)\Delta E + (hu - v)q + \tfrac{1}{2}(hu - v)\Delta q + \tfrac{1}{2}q(h\Delta u - \Delta v) \\ + \tfrac{1}{2}qu\Delta h + \Delta M = \mu(\Delta E + a\Delta l + l\Delta a), \end{aligned} \tag{19'}$$

$$\Delta\bar{u} = \frac{\frac{1}{2}\Delta v_{-3} + 1\frac{1}{2}\Delta v_{-2} + 2\Delta v_{-1} + \Delta v}{2hT} - \frac{1}{2h^2T}\Delta h(\tfrac{1}{2}v_{-3} + 1\tfrac{1}{2}v_{-2} + 2v_{-1} + v). \tag{21'}$$

The unknowns of this system are Δa, Δu, Δv, Δv_c, ΔE, Δp, Δq, and $\Delta\bar{u}$; the independent variables: Δl, $\Delta\bar{g}$, $\Delta g'$, Δh, ΔM. The coefficients in these equations are all magnitudes relating to the actual situation and to some extent represent the economic structure as far as it seems to be important for long-run changes.

V. STATISTICAL INFORMATION

We have attempted to measure approximately the values of the coefficients in equations (14) to (21) and (17′), (19′), and (21′). This

required a considerable amount of statistical work, the details of which would take far too much space to be given here. Of course great accuracy cannot be claimed for the results obtained. Some further trials have shown, however, that the character of most of the results obtained does not change very much, if the statistical values taken are replaced by different values based on uncertainty margins estimated. All this must, however, be preserved for a subsequent monograph.

Before mentioning the figures used something must be said on the system of units used. For both time points considered the following principles for the choice of units have been applied:

(i) All prices have been taken equal to one;

(ii) All money amounts have been measured in total wages bill as unit.

TABLE 1

SUMMARY OF STATISTICAL INFORMATION USED

Symbol	Description	Value 1910	Average value 1919–1932‡
$\bar{g}$	Mean labor quota in consumers'-goods industries for normal production	0.40	0.54
γ	Increase of marginal labor quota in consumers'-goods industries per unit of increase in output	0.11	0.13
u	Volume of production of consumers' goods	1.83	1.51
$\bar{u}$	"Normal" volume of production of consumers' goods	1.83	1.51
g'	Labor quota in investment-goods industries	0.75	0.70
v	Volume of production of investment goods	0.34	0.37
h	Deflated* depreciation allowance per unit of output	0.07	0.10
ϵ	Marginal propensity to consume for nonworkers	0.70	0.70
E''	Amount of total savings	0.22	0.21
μ	Transition period† (unit: 10 years)	0.05	0.07
$\bar{n}$	"Normal" remuneration of nonworkers per unit of output of consumers' goods†	0.43	0.26
n'	Remuneration of nonworkers per unit of output of investment goods	0.25	0.20
T	Lifetime of investment goods (unit: 10 years)	2.5	2.5
v_{-3}	Volume of production of investment goods in time period −3	0.07	0.10
v_{-2}	Volume of production of investment goods in time period −2	0.12	0.24
v_{-1}	Volume of production of investment goods in time period −1	0.18	0.36

* I.e., depreciation allowance if prices of consumers' goods and of investment goods are taken equal to one.

† For explanation of term, see text.

‡ The principle underlying the choice of units invalidates, in some respects, the comparability of the last column with the last but one.

From (i) it follows that—at the moment considered—each quantity figure (a, u, v, etc.) is equal to the corresponding value figure (L, up, vq, etc.). From (ii) it follows that—at that same moment—$a=1$, since $L=1$ and $l=1$. Of course this is not necessarily the case for any later moment, since all variables considered may change.

Given this system of units, certain comparisons between the 1910 and the 1919–1932 figures are not possible. The values $u=1.83$ for 1910 and $u=1.51$ for 1919–1932 do not mean, e.g., that the volume of production fell. They mean that the value of production of consumers' goods fell in proportion to total wages.

The figures used are given in Table 1. The reader will easily find all he wants for substituting in the equations (14)–(21). An exception must be made for the values Δv_{-3}, Δv_{-2}, and Δv_{-1}. These are, however, only contributing to the nonvariable terms in the equations which do not interest us for our problem and which have, therefore, not even been calculated.

One general remark may be added. The aim of statistical measurement has not been to test the equations (1) to (12). On the contrary, these have been considered as generally accepted and a number of the structural coefficients have been calculated with their help. Only equation (13) will be found not to be satisfied; the values found for the right-hand member and the left-hand member are rather divergent. There may be good reasons for this. Anyhow, the consequences of this discrepancy for our results have been calculated and were found to be unimportant.

VI. RESULTS OF CALCULATIONS

Putting in the figures and solving for the unknown yields the results given in Table 2.

The first line of Table 2 means:

Case 1: $\Delta a = -0.55\Delta l + 0.97\Delta \bar{g} + 0.11\Delta g' + 2.18\Delta h + 9.42\Delta M +$ constant, and so on.

The application of these results to concrete problems requires some caution in that often changes in one of the independent variables may entail changes in others. This must be considered carefully for each case treated.

A. Technical Progress and Employment.

Technical progress may be taken to mean any change in technical coefficients yielding lower costs per unit of product than before. Disregarding for a moment g', the reduction may be the result of:

(i) a reduction in $\bar{g}$, with an accompanying (but smaller) increase in h(l and q are supposed to be one); commonly known as *mechanisation*;

(ii) a reduction $\bar{g}$ without a change in h, mostly known as *rationalisation*; and

(iii) a reduction in h with or without a (smaller) increase in $\bar{g}$. Such cases will frequently represent what Schumpeter called "*new combinations.*"

TABLE 2

RESULTS OF CALCULATIONS

Coefficients obtained in solution for left-hand variable.

Case		Δl	$\Delta \bar{g}$	$\Delta g'$	Δh	ΔM
1	$\Delta a =$	−0.55	+0.97	+0.11	+ 2.18	+ 9.42
2		−0.92	+0.59	+0.29	+ 1.08	+ 8.72
3		−1.03	+0.59	+0.13	+ 1.49	+ 8.72
1	$\Delta u =$	−0.79	−1.37	−0.20	− 3.94	+14.96
2		−1.13	−1.17	+0.078	− 2.87	+11.04
3		−1.29	−1.17	−0.13	− 4.46	+11.04
1	$\Delta v =$	−0.35	−0.38	−0.28	+ 0.89	+ 4.20
2		−0.36	−0.33	−0.24	+ 0.99	+ 3.10
3		−0.39	−0.33	−0.28	+ 2.54	+ 3.10
1	$\Delta E =$	−0.40	−0.96	−0.20	− 2.26	+10.48
2		−0.44	−0.60	−0.012	− 1.12	+ 5.70
3		−0.52	−0.60	−0.12	− 1.55	+ 5.70
1	$\Delta p =$	+0.52	+0.91	+0.093	+ 2.48	+ 0.98
2		+0.60	+0.88	+0.14	+ 2.10	+ 1.11
3		+0.59	+0.88	+0.12	+ 3.23	+ 1.11
1	$\Delta q =$	+0.75	—	+1.00	—	—
2		+0.70	—	+1.00	—	—
3		+0.78	—	+1.11	+ 1.11	—
1	$\Delta v_e =$	−0.35	−0.38	−0.28	+ 0.89	+ 4.20
2		−0.36	−0.33	−0.24	+ 0.99	+ 3.10
3		−0.39	−0.33	−0.28	+ 2.54	+ 3.10
1	$\Delta \bar{u} =$	−1.00	−1.11	−0.82	−34.82	+12.14
2		−0.58	−0.53	−0.38	−22.60	+ 5.01
3		−0.62	−0.53	−0.45	−26.23	+ 5.01

1. Prewar case [equations (14)–(21) with primes where they exist].
2. Comparable postwar case (same equations, but postwar figures).
3. Complete postwar case (equations without primes).

The remarkable result obtained by our calculations is that reductions in $\bar{g}$ (increases in labor productivity) are unfavorable to employment. This stands in contrast to what is known as the compensation theory. Let us go into some more detail here.

Since $a = gu + g'v$, where g is the amount of labor per unit of con-

sumers'-goods output, the direct consequence of a change in g may be taken to mean $\Delta a = u\Delta g$, the change in a for constant u, which, under these circumstances, is equal to $u\Delta\bar{g}$ or $1.83\Delta\bar{g}$ in the prewar case and $1.51\Delta\bar{g}$ in the postwar case. The compensation theory (whatever form it be given) holds that this direct, unfavorable, influence is offset by indirect consequences, which evidently result in changes in production. A number of these indirect consequences are taken account of in our calculations, as, e.g., price change, influence of change in incomes, etc. Our results show that these repercussions are not able to compensate for more than about 50 per cent of the direct influence.

Our results do not include repercussions via the other independent variables. But they could be made to do so if we knew how much l, M, etc., change for a given change in $\bar{g}$. There is little reason to include changes in l. Most authors are interested in knowing whether there will be compensation without wage changes.

Changes in h and in M may, however, be included. But it is not easy to see what relation exists between a given change $\Delta\bar{g}$ in $\bar{g}$, and the changes in h and M that accompany them.

As to changes in h only a certain limit can be indicated: it is in the nature of technical progress that $\Delta\bar{g}+\Delta h<0$, since the left-hand side represents the increase in cost of production per unit. This does not, however, in our case give very narrow limits as to the results of changes in h. If $\Delta h = -\Delta\bar{g}$ (one extreme) we find that full compensation would be obtained, since in Δa the coefficient for $\Delta h >$ that for $\Delta\bar{g}$. If, on the other hand, $\Delta h = 0$, which is certainly within the limit of possibilities, our previous conclusion still holds. From this it seems that the consequences of technical progress on employment are widely divergent for various types of technical changes. It may therefore be useful to know something on the actual changes in $\bar{g}$ and h. Our—admittedly very rough—estimates for the United States as a whole over the period 1850–1910 suggest that there is not a very close relation between $-\Delta\bar{g}$ and Δh, and, as far as such a relation exists, $-\Delta\bar{g}$ is about ten times as large as Δh. This would be somewhat reassuring, since it would mean that the influence of changes in h on employment is not so large.[2]

There remains the question of the repercussion on M. It is equally difficult to see of what nature and extent this repercussion is. It could be argued that an increase in labor productivity stimulates new investment activity and therefore ΔM. This connection is not, however,

[2] Similar calculations were made for the Netherlands. The coefficient found for $\Delta\bar{g}$ was very near to that found for the U. S., but the coefficient found for Δh was much smaller. The difficulty just dealt with did not exist therefore for that country.

necessarily very intimate. It depends on the character of the technical change. An increase in labor productivity may, but need not, be accompanied by an increase in real investment. Therefore it seems better to hold separated the two phenomena and to state explicitly that *our conclusions concerning partial compensation bear on the case where no additional investment occurs as a consequence of the change in g.*

B. Wages and Employment: the Elasticity of the Demand for Labor.

Our formulae enable us to find, as a by-product, what influence on employment is exerted by a change in wage rates. By the choice of our units the coefficients for Δl found in the equations for Δa are equal to the elasticity of demand for labor. This elasticity would be somewhat more than one-half for 1910 and about unity for the postwar period. Some qualifications must, however, be kept in mind. First, that we are dealing with long-run elasticities and that our figures do not take account of such cyclic phenomena as hoarding in depression. Secondly, that they have been made under the hypotheses enumerated, of which the most important one is that no changes in the other independent variables occur as a consequence of the wage-rate change. This means in particular that labor productivity would not be affected by a change in wages. It seems more realistic, at least for the very long run (after a couple of years, e.g.), to assume that the technical constants $\bar{g}$, g', h are functions of the wage rate. It is not easy to get accurate information on these functions which, by the way, must depend on the production function. A very rough estimate, based on a study by Professor Gustav Åkerman[3] may be made in the following way. Professor Åkerman found that, out of 19 cases of rationalisation which he studied, $12\frac{1}{2}$ were due to increases in real wages (cases which were described as only partly due to increases in real wages being counted for one-half), whereas out of these $12\frac{1}{2}$ cases, $6\frac{1}{2}$ would, in the case of a wage reduction, be undone again. There is, therefore, a clear indication of "hysteresis": a different reaction for $\Delta l > 0$ and $\Delta l < 0$. We may summarize the situation by saying that 0.5 ± 0.2 of the cases of rationalisation were due to wage changes, where the upper sign relates to wage rises and the lower to wage falls. Now $\Delta \bar{g}$ was, between 1921 and 1931, in our units, equal to -0.14, whereas $\Delta(l-p)$, representing the change in real wage rate, amounted to $+0.22$. If, since we have to do with a rise in wage rates, 7/10 of the fall in $\bar{g}$ or -0.10 is to be attributed to the change $+0.22$ in $l-p$, then the relation between $\Delta \bar{g}$ and $\Delta(l-p)$ must be

[3] "Om den industriella rationaliseringen och dess verkningar," *Arbetslöshetsutredningens betänkande I*, Bilagor, Band 2, Stockholm 1931.

$$\Delta\bar{g} = -\frac{0.10}{0.22}\Delta(l - p) + \Delta\bar{\bar{g}}$$

for wage rises, where $\bar{\bar{g}}$ is the part of $\bar{g}$ which is to be attributed to other factors than wages. For wage falls the coefficient has to be changed in the proportion 0.7 to 0.3. We therefore get:

$$\bar{g} = -0.45\Delta(l - p) + \Delta\bar{\bar{g}} \text{ for } \Delta(l - p) > 0$$

$$\bar{g} = -0.19\Delta(l - p) + \Delta\bar{\bar{g}} \text{ for } \Delta(l - p) < 0,$$

or summarized: $\Delta\bar{g} = -(0.32 \pm 0.13)\Delta(l - p) + \Delta\bar{\bar{g}}.$

This may now be combined with our results for case 3: $\Delta a = -1.03\Delta l + 0.59\Delta\bar{g}$, neglecting further terms, and $\Delta p = 0.59\Delta l + 0.88\Delta\bar{g}$, neglecting further terms.

It follows that

$$\Delta\bar{g} = -(0.32 \pm 0.13)[\Delta l - 0.59\Delta l - 0.88\Delta\bar{g}] + \Delta\bar{\bar{g}},$$

or

$$\Delta\bar{g} = -(0.17 \pm 0.04)\Delta l + (1.4 \pm 0.2)\Delta\bar{\bar{g}}.$$

Finally,

$$\Delta a = -1.03\Delta l - (0.10 \pm 0.02)\Delta l + (0.83 \mp 0.12)\Delta\bar{\bar{g}}.$$

The elasticity of demand for labor would, according to this rough evaluation, not be changed considerably by the reaction on $\bar{g}$ which is exerted by l. And in this correction the influence on h, which will generally be of the opposite sign, has even been neglected.

C. Hours and Employment: the Influence of a 40-Hour Week on Employment.

We are also able to find the influence of a change in working hours from, say, 48 to 40. Taking a week as the unit of labor, this means that $\bar{g}$ and g' will rise in the proportion 5:6; thus $\Delta\bar{g} = 0.11$ and $\Delta g' = 0.14$. The effect on h is not certain; if depreciation were proportional to production, no change in h would be involved; if it were proportional to time, a maximum change in h of 1/5 or 0.02 would be the effect of the change in hours; therefore $\Delta h = 0.01 \pm 0.01$. As to wages, two different cases may be considered; first, no change in weekly wages which means that $\Delta l = 0$; secondly, a proportionate reduction in weekly wages, meaning that $\Delta l = -0.17$. Using formula (3) we find:

Δa, for:	$\Delta h = 0$	$\Delta h = 0.02$
$\Delta l = 0$	0.08	0.10
$\Delta l = -0.17$	0.26	0.28

It must again be emphasized that these changes represent long-run

changes, disregarding cyclic influences. They seem to be very favorable for the case of shorter hours: an increase of 8 to 10 per cent of employment would result when weekly wages are kept constant and one of 26 to 28 per cent if hourly wages are kept constant.

It will be clear that the effect on total consumption is less favorable. Using the formula for u (case 3) we find:

Δu, for:	$\Delta h=0$	$\Delta h=0.02$
$\Delta l=0$	-0.15 (-10%)	-0.23 (-15%)
$\Delta l=-0.17$	$+0.07$ ($+5\%$)	-0.02 (-1%)

The percentage changes have been given in brackets ($u=1.51$).

D. Concluding Remarks

We have not yet exhausted our formulae. They enable us, in principle, to calculate consequences of other structural changes and also to calculate the effects on the other variables such as p, q, E, etc. Part of this may be left to the reader and to later publications. As an example, one further problem may be considered, viz., to find the increase in total incomes $al+E$ or $\Delta a+\Delta E$ for a given increase in M, obtained by additional investments ΔM financed by credit creation. Evidently this is the problem of the multiplier, but under conditions somewhat different from those assumed by Kahn and Keynes. A reserve capacity has been assumed to exist in this sense that less and less "good" investment goods are available for increases in production [cf. equations (3) and (8)]. No dole has been supposed to exist and the community considered is a closed one. From Table 2 we find: $\Delta a+\Delta E = (8.72+5.70)\Delta M$, which means a multiplier of about 14.

Similar calculations[4] have been made for Holland; they show, in many respects, similar results; but the multiplier is found—as it should be—to be much lower, viz., of the order of magnitude of 2.

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[4] These will be published in a Report made by the Netherlands Central Statistical Office on the request of the High Labor Council.

SOME PROBLEMS IN THE MEASUREMENT OF INCOME ELASTICITIES*

By H. Gregg Lewis and Paul H. Douglas

Nine years ago one of the authors began a series of investigations into the relationship between changes in family and individual income and changes in expenditures upon specific groups of commodities. This utilized the family budget data collected in three large studies, namely: (1) The 1901 study by the Bureau of Labor on the income and expenditures of some 11,000 families.[1] (2) The 1918–19 investigation by the Bureau of Labor Statistics which covered slightly over 12,000 families.[2] (3) The study of 2,866 white farm families which was made in 1922–24 by the U. S. Department of Agriculture.[3]

While the completion of this work was delayed for some years, it has been resumed with the co-operation of several associates[4] during the last two years, and it is planned to publish the results shortly. Comparisons between these results and those obtained by the Consumer Purchases Study will, of course, be of great interest.

Before discussing certain problems that arise in the inductive analysis of consumer expenditure patterns, it may be well to outline as briefly as possible some elementary theoretical concepts. These concepts are not new.[5] What we shall attempt to do here is to bring them together, and relate them to the geometry of cost and revenue curves.

For the purposes of immediate discussion we shall assume that we can observe an individual or family (consumer unit) varying his rate of expenditure per unit period as his income changes in a situation in which everything is held constant except income and expenditure.

THE TOTAL-EXPENDITURE CURVES

Let us denote the several commodity groups—food, clothing, shelter, etc., but not including savings—by the symbols

$$(G_1), (G_2), \cdots, (G_i), \cdots, (G_n),$$

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[1] *Eighteenth Annual Report of the U. S. Commissioner of Labor* (1903), "Cost of Living and Retail Prices of Food," pp. 298–633.

[2] U. S. Bureau of Labor Statistics, *Cost of Living in the United States.* Bulletin 357, 1924, 466 pp.

[3] E. L. Kirkpatrick, *The Farmer's Standard of Living.* U. S. Department of Agriculture, Department Bulletin No. 1466. 63 pp.

[4] Notably Mrs. Frances G. McIntyre and Miss Yetta Abend.

[5] See especially R. G. D. Allen and A. L. Bowley, *Family Expenditure,* London, 1935.

the corresponding total money expenditures, including imputed money expenditures, of an individual or family by

$$x_1, x_2, \cdots, x_i, \cdots, x_n,$$

and individual or family income (including imputed income) by Y. The total *monetary consumption* then is

$$(1) \qquad X \equiv \sum_{i=1}^{n} x_i,$$

and total savings,

$$(2) \qquad x_s \equiv Y - X \equiv Y - \sum_{i=1}^{n} x_i.$$

Under the ideal assumption stated above, the total money expenditures on the various groups will depend on income alone; we shall represent these functional relations by the equations

$$(3) \qquad x_i = f_i(Y), \qquad (i = 1, 2, 3, \cdots, n).$$

These are the equations of the total-group-expenditures or group-budget curves.

The equation of the total-budget curve or total-consumption curve then is

$$(4) \qquad X = \sum_{i=1}^{n} f_i(Y) \equiv F(Y),$$

and the equation of the savings curve,

$$(5) \qquad x_s = Y - F(Y) \equiv f_s(Y).$$

These total-group-expenditure curves are analogous to the familiar total-cost and total-revenue curves.

THE AVERAGE-EXPENDITURE CURVES

Corresponding to each total-expenditure curve there is an average-expenditure curve, described by the equation

$$(6) \qquad \bar{x}_i = \frac{x_i}{Y} = \frac{f_i(Y)}{Y},$$

where $\bar{x}_i$ is the average expenditure for the group (G_i). The average expenditure corresponding to any income measures the proportion of that income that is spent on the commodity group in question.

The average-consumption curve, which, in Keynes' terminology, is the average-propensity-to-consume curve, is given by the equation

$$\overline{X} = \frac{F(Y)}{Y}, \tag{7}$$

and the average-propensity-to-save curve by the equations

$$\bar{x}_s = \frac{f_s(Y)}{Y} = 1 - \frac{F(Y)}{Y} \cdot \tag{8}$$

Geometrically, the average expenditure corresponding to any point on the total-expenditure curve is given by the slope of the radius vector through that point.

In his studies,[6] Engel was concerned only with the relation between average expenditure and income. His familiar law states that the proportion of total income spent for food declines as income increases; that is, that average expenditure for food decreases with income. More recently, however, we have become necessarily interested not only in expenditures on the average, but in expenditures at the margin as well.

THE MARGINAL-EXPENDITURE CURVE

The marginal-expenditure curves are described by the equations

$$m_i = \frac{dx_i}{dY} = f_i'(Y), \qquad (i = 1, \cdots, n), \tag{9}$$

where m_i denotes the marginal expenditure for the group (G_i). These curves are analogous to the familiar marginal-revenue and marginal-cost curves.

The marginal expenditure corresponding to any income measures the absolute change in total expenditure corresponding to a small increase over that income; it measures the proportion of an additional dollar of income that would be spent on the commodity group in question. The marginal-consumption curve, or marginal-propensity-to-consume curve, is given by the equation

[6] Ernst Engel, "Die Produktions und Consumtionsverhältnisse des Königreichs Sachsen." Originally published in the *Zeitschrift des Statistischen Bureaus des Königlich Sächsischen Ministeriums des Innern,* November 22, 1857 and reprinted in Engel, *Die Lebenkosten Belgischer Arbeiter-Familien,* Dresden, 1895, 54 pp., and *Bulletin de l'Institut International de Statistique,* Tome IX, 1895. Some of this material was also given by Carroll D. Wright in the 1875–76 *Report of the Massachusetts Bureau of Statistics of Labor,* pp. 443 ff., in which Mr. Wright stated that Engel had four "laws" of expenditure, namely, that the average percentage spent on food declined as income increased, that the average for sundries increased, and those for clothing and rent remained constant. There is no mention of the last three conclusions in Engel's monographs but they may have been stated to Mr. Wright by Dr. Engel in private correspondence.

$$M = \frac{dX}{dY} = F'(Y) \tag{10}$$

and the marginal-propensity-to-save curve by the equation

$$m_s = \frac{dx_s}{dY} = 1 - M = f_s'(Y). \tag{11}$$

Geometrically, the marginal expenditure corresponding to any point on the total-expenditure curve is given by the slope of the curve at that point. Marginal expenditure is increasing when the total-expenditure curve is concave upward and is decreasing when the total-expenditure curve is concave downward. Marginal expenditure is greater than, less than, or equal to average expenditure according as average expenditure is increasing, decreasing, or constant.

THE INCOME ELASTICITY OF EXPENDITURE

At any point on the total-expenditure curve the elasticity of total expenditure with respect to income or *income elasticity of expenditure,*[7] ϕ, is defined as the ratio of the proportional change in total expenditure to a proportional increase of income when the increase of income is very small. The total-expenditure curve is elastic when the ratio is greater than unity, and inelastic when the ratio is less than unity. Analytically, the curve of the budgetary elasticity is given by the equation

$$\phi_i = \frac{dx_i}{dY}\,\frac{Y}{x_i} = \frac{d \log x_i}{d \log Y}\,. \tag{12}$$

Equation (12) may be written in the form

$$\phi_i = \frac{\dfrac{dx_i}{dY}}{\dfrac{x_i}{Y}} = \frac{m_i}{\bar{x}_i}\,. \tag{13}$$

Interpreted in this way, the total-expenditure curve is elastic or inelastic according as the proportion of income spent at the margin on the commodity group is greater than or less than that spent on the average. Thus, ϕ is greater than, less than, or equal to unity, according as average expenditure is increasing, decreasing, or constant.

When the total-expenditure curve is plotted on logarithmic scales,

[7] Hereafter we shall refer to the *income elasticity of expenditure* simply as the *income elasticity.*

the income elasticity may be measured graphically as the slope of the curve.

Inasmuch as we are assuming prices to be constant, the income elasticity is the same thing as the income elasticity of demand.[8]

THE ELASTICITY OF THE AVERAGE-EXPENDITURE CURVE

The elasticity of the average-expenditure curve is a concept analogous to that of flexibility of price (the reciprocal of elasticity of demand). It is defined for the various groups by the equations

$$\mu_i = \frac{d(x_i/Y)}{dY} \cdot \frac{Y}{\frac{x_i}{Y}} = \frac{d\bar{x}_i}{dY} \cdot \frac{Y}{\bar{x}_i} = \frac{d \log \bar{x}_i}{d \log Y}, \tag{14}$$

where μ is the elasticity.

The elasticity of average expenditure bears the following simple relation to the income elasticity:

$$\mu_i = \phi_i - 1. \tag{15}$$

Thus, knowing one of these elasticities, we immediately have the other by adding or subtracting unity as the case may be.

The elasticity of average expenditure is related to marginal expenditure in the following manner:

$$m_i = \bar{x}_i(\mu_i + 1). \tag{16}$$

Marginal expenditure is greater than, less than, or equal to zero, according as the elasticity of average expenditure is greater than, less than, or equal to -1; or, from equation (15), as the income elasticity is greater than, less than, or equal to zero.[9]

THE ELASTICITY OF THE MARGINAL-EXPENDITURE CURVE

The elasticities of marginal expenditure for the various groups are given by the equations

[8] Let Q_i be the composite quantity, and P_i the composite price of the commodity or commodity group G_i.

Then we have

$x_i = P_i Q_i$ and

$$\frac{dx_i}{dY} \cdot \frac{Y}{x_i} = \frac{P_i dQ_i}{dY} \cdot \frac{Y}{P_i Q_i} = \frac{dQ_i}{dY} \cdot \frac{Y}{Q_i}, \text{ since } P_i \text{ is constant.}$$

[9] Equation (16) is analogous to the familiar relation: Marginal revenue

$$= \text{price}\left(1 - \frac{1}{\text{elasticity of demand}}\right)$$

$$(17)\qquad \nu_i = \frac{dm_i}{dY}\cdot\frac{Y}{m_i} = \frac{Yd^2x_i}{dY^2}\cdot\frac{dY}{dx_i} = \frac{d\log m_i}{d\log Y}\,.$$

At any point on the marginal-expenditure curve this elasticity measures the ratio of the proportional change in marginal expenditure to a proportional increase in income, when the increase of income is very small.

The elasticity of marginal expenditure is related to the income elasticity by the equation

$$(18)\qquad \nu_i = \phi_i - 1 + \frac{d\phi_i}{dY}\cdot\frac{Y}{\phi_i}\,.$$

That is, the elasticity of marginal expenditure is one less than the sum of the income elasticity and the elasticity of the income elasticity. Equations (18) are also useful in indicating whether the income elasticity is increasing or decreasing. For ϕ_i greater than zero, the income elasticity increases, decreases, or remains constant according as $\phi_i - \nu_i$ is less than, greater than, or equal to unity. For ϕ_i less than zero, ϕ_i increases, decreases, or remains constant according as $\phi_i - \nu_i$ is greater than, less than, or equal to unity.

In general, the more urgent the expenditure on a commodity group:

(1) The larger the expenditure intercept of the total-group-budget curve; that is, the greater the expenditure on the commodity group when income is zero.

(2) The smaller the income elasticity and, therefore, the elasticity of average expenditure.

(3) The smaller the elasticity of marginal expenditure.

CHOICE OF TYPE OF EQUATION

If the foregoing analysis is to be useful in actual inductive study of family expenditures, it is desirable that the type of total-expenditure curve satisfy at least the following conditions:

(1) The curve should be of such form that it need not pass through the origin.

(2) The marginal expenditure, dx_i/dY, should not be constant. Preferably, of course, the type of curve should also satisfy the condition that the second derivatives and elasticities of x_i and $\bar{x}_i$ should not be constant.

If total money consumption, X, rather than income, Y, is taken as the independent variable in fitting the budget curves, condition (1)

is replaced by the restriction that the curve *must* not have a positive expenditure intercept.[10]

We have introduced these minimum conditions to insure a minimum of flexibility in the analysis of the budget curves. For example, if we fit straight lines it is impossible to study any change of marginal expenditure with respect to changes in income.

There are, moreover, certain economic considerations which inveigh heavily against the fitting of some types of curves. This is especially true of the straight line, a type which has been much used. In the extensive studies of Allen and Bowley,[11] for example, the straight line was used exclusively, and Engel's law was restated as the "law of linearity." We shall not take time here to examine their data or some of their questionable justifications for fitting the straight line where they found curvilinearity.

It is important, however, to realize the properties of the so-called "law of linearity," described by the equation

$$x = a + bY. \tag{19}$$

The average-expenditure curve is given by the equation

$$\bar{x} = \frac{a}{Y} + b, \tag{20}$$

the marginal expenditure by

$$m = b, \tag{21}$$

and the income elasticity by

$$\phi = \frac{b}{\dfrac{a}{Y} + b}\cdot \tag{22}$$

The elasticity of marginal expenditure is obviously zero, so that we have,

$$\mu = -\text{elasticity of } \phi. \tag{23}$$

That is, one per cent increase of average expenditure is accompanied by one per cent decline of the income elasticity.

Chart 1, Figures A, B, and C respectively illustrate the three cases: *a*, the expenditure-intercept constant, greater than zero, equal to zero,

[10] If we are not interested in studying the expenditure for very low incomes, conditions relating to the behavior of the budget curves for low incomes are not important.

[11] *Op. cit.*

and less than zero.[12] In these three figures the curves have been plotted on an arithmetic scale.

In all cases marginal expenditure is constant. When *a* is greater than zero (Figure A), average expenditure is greater than marginal expenditure, but decreases, approaching marginal expenditure as income increases. Since the income elasticity is equal to the ratio of marginal expenditure to average expenditure, the income elasticity is less than unity at all points. Moreover, since marginal expenditure is constant and average expenditure decreasing, the income elasticity increases as income increases, approaching unity. The average-expendi-

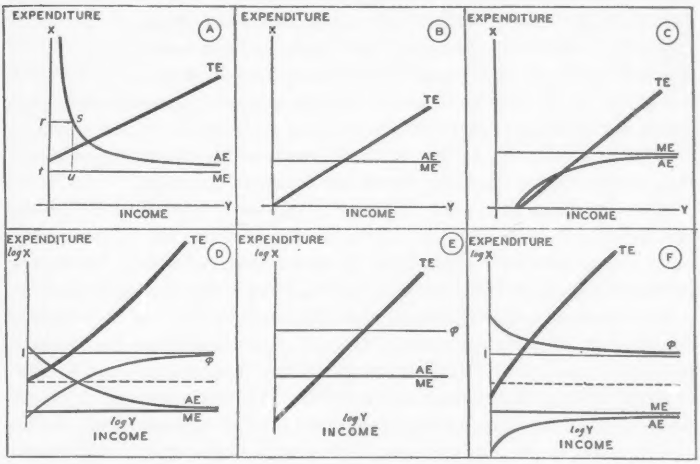

CHART 1.—Three cases of straight-line expenditure curves, diagrammed on arithmetic and logarithmic scales.

ture curve is a rectangular hyperbola when the marginal-expenditure curve is taken as the horizontal axis, and the area of any rectangle, *rstu*, between the average and marginal-expenditure curves is equal to the constant *a*.

When *a* is equal to zero (Figure B), average and marginal expenditure are constant and equal to each other, and the income elasticity then constant and equal to unity.

When *a* is less than zero (Figure C), the average-expenditure curve

[12] The vertical scale unit for the average and marginal expenditures is larger than that for total expenditure. On these charts TE, ME, and AE denote the total-, marginal-, and average-expenditure curves respectively, and ϕ the income-elasticity curve.

is a rectangular hyperbola *under* the marginal-expenditure curve. Average expenditure increases approaching marginal expenditure as income increases. Thus the income elasticity is at all points greater than unity, but decreases, approaching unity, as income increases.

It is obvious that the income elasticities for the various groups will not be much different in size at high income levels, all of them being close to unity.

Chart 1, Figures D, E, and F, illustrate on the logarithmic scale the three cases shown respectively in Figures A, B, and C. It will be noted that when a is greater than zero (Figure D), the total-expenditure curve is concave upward on a log scale with slope less than unity, but approaching unity as income increases. Since the slope of the total-expenditure curve on the log scale is equal to the income elasticity, the upward concavity indicates an increasing income elasticity approaching unity. It will also be observed that the slope of the average-expenditure curve is the negative of the slope of the income-elasticity curve, indicating that average expenditure changes at the same proportionate rate as the income elasticity but in the opposite direction.

When a is less than zero (Figure F), the total-expenditure curve is concave downward with slope greater than unity, but approaching unity as income increases. That is, the income elasticity is always greater than unity, but decreases, approaching unity for high incomes.

The most serious criticism of the "law of linearity" is the obvious fact that it requires constant marginal expenditures on the various commodity groups. Each additional dollar of total income will be distributed among the various commodities and services in proportions which will be constant irrespective of the level of income to which the additions are made. For example, if the value of the marginal expenditure, b, in the case of food is 0.30, this would mean that a decrease of \$100 in the income of a man with \$10,000 a year would decrease his food expenditures by \$30; while the transfer of the same \$100 to a man with an income of \$1,000 would merely increase his food expenditures by the same \$30. Thus changes in the distribution of income between income classes effect no changes in the distribution of expenditures. In other words, under the formula, the only factor which determines the amounts spent on the various commodity groups is the total income of the consumer population studied.

Even if the major proportion of the total income of the population were concentrated in the hands of those in the high income brackets, while the remainder lived in great poverty, the same total amounts would be spent for food, clothing, shelter, etc., on the one hand, and for champagne and polo ponies on the other, as in a population characterized by equality of income.

This is, of course, manifestly not the case, at least over any significant income range, and especially not true for the low and high income ranges. There is, on the contrary, every presumption that the marginal expenditures for people with high incomes will be smaller in the case of necessities than those for people with low incomes, and larger in the case of luxuries. The use of the straight-line formula prevents the exploration of any such tendency.

A less important criticism of the use of the straight line is that it requires decreasing income elasticities for those commodity groups classed as luxuries—that is, for groups having income elasticities greater than unity—and increasing income elasticities for necessities—those with income elasticities less than unity. Thus under this formula, the distinction between "luxuries" and "necessities" rapidly diminishes as income increases, until for high incomes all commodity groups have income elasticities close to unity.

Moreover, the larger the income elasticity, the greater is its rate of decrease, or, what is the same thing, the smaller the income elasticity, the greater is its rate of increase. This surely is unreasonable, at least for commodity groups of very low or very high elasticities. The more "necessary" a commodity is the more rapidly it approaches the reasonable income-elasticity characteristics of a luxury, and vice versa.

The straight line has in its favor, of course, its computational simplicity; and, if the income range is short, and does not include high or low income ranges, and if we are interested only in the expenditure behavior at or near the mean income of the population studied, the straight line gives about as satisfactory results as any other type of curve. But if we want to study the expenditure behavior over a significant income range, then it is desirable to fit curves less restricted than the straight line.

A type of curve which is simple to compute, easy to understand, and somewhat less restricted than the straight line is the logarithmic straight line or constant-elasticity curve

$$x = ay^b, \tag{24}$$

which in logarithmic form is

$$\log x = \log a + b \log y. \tag{25}$$

The average expenditure is

$$\bar{x} = ay^{b-1}; \tag{26}$$

the marginal expenditure,

$$m = aby^{b-1}; \tag{27}$$

the income elasticity,

$$\phi = b; \tag{28}$$

and the elasticities of marginal and average expenditure,

$$\nu = \mu = b - 1. \tag{29}$$

It will be noted that the income elasticity is constant and equal to b, and the elasticities of marginal and average expenditure both equal to $b-1$. This constancy of the elasticities is, obviously, a disadvantage, but not so serious as the perverse elasticity restrictions on the straight line.

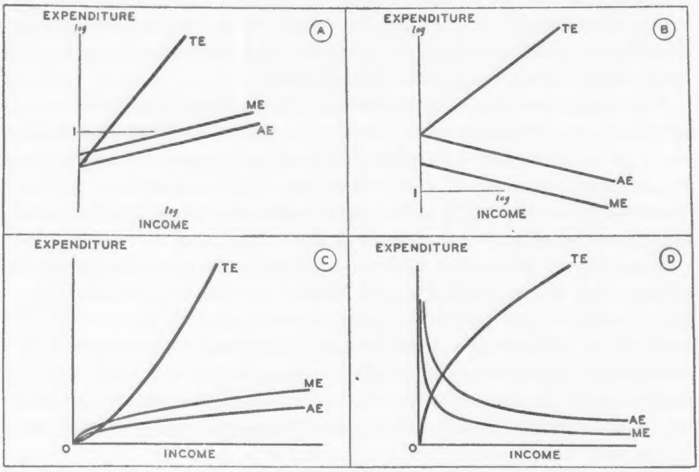

CHART 2.—Two cases of constant-elasticity total-expenditure curves, diagrammed on logarithmic and arithmetic scales respectively.

Chart 2 illustrates two cases of logarithmic straight lines, Figures A and C showing b greater than unity, and B and D showing b less than unity. When b is equal to unity the case becomes that of a straight line passing through the origin. On Figures A and B the curves have been plotted on the logarithmic scale, and on C and D on the arithmetic scale.

Since the total-, average-, and marginal-expenditure curves have constant elasticities, they are all linear on the logarithmic scale. When b is greater than unity (Figure A), both marginal and average expenditures increase with income; when b is less than unity (Figure B) both decline with income.

On the arithmetic scale the total-expenditure curve is concave upward when b is greater than unity, and concave downward when b is less than unity; indicating that marginal expenditure increases or decreases according as b is greater than or less than unity. The average- and marginal-expenditure curves are concave upward if b is greater than 2 or less than 1, and concave downward if b is between 1 and 2.[13]

The constant-elasticity curve is not subject to the main criticism we have advanced against the straight line, namely, that it perforce requires that redistributions of income would not affect the expenditure distribution. The logarithmic straight line implies that a redistribution in favor of those with low incomes would increase the total expenditure of the population on necessities, and decrease the total expenditure on luxuries. This follows from the fact that marginal expenditure increases or decreases with income according as the income elasticity is greater than or less than unity.

When income is taken as the independent variable, however, the logarithmic straight line suffers from the restriction that it must pass through the origin, so that in the low income ranges it will not be very accurate.

SOME INTERPRETATIVE PROBLEMS

Most of the preceding discussion of the analysis of expenditure patterns has been confined to the ideal situation.

Rigorously, however, what we usually observe is not the pattern of *change* in the rates of expenditure which a family follows as its income changes in the ideal situation, but the static picture of the rates of expenditure of *different* people with *different* incomes living in more or less *different* situations. That is, the expenditure curves are cross sections *at a given time* for *different* people, and show the mean expenditures corresponding to different income levels.

[13] Chart 2, Figure C illustrates the case where b is between 1 and 2; Figure D,

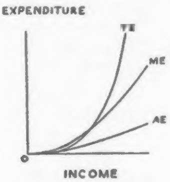

CHART 3.

the case where b is between 0 and 1. The accompanying Chart 3 illustrates the case where b is greater than 2, and thus both average- and marginal-expenditure curves are concave upward.

Thus, in our inductive analysis of budget data, certain important time considerations are involved, which are concerned with the interpretation of the expenditure curves as patterns of *change.*

A family's or individual's *present* rates of expenditure depend not only on their present income and social situation, but also on:

(1) The length of time they have had their present income and have lived in their present social-economic situation, and

(2) Their whole former income and social-economic history.

These time considerations are very important in interpreting the expenditure curves, and derivative concepts. An example will perhaps make this clear. Suppose that the expenditure curve for food indicates that a family with a present income of $2,000 a year spends on the average $800 for food, while a family with an income of $2,100 spends $830 for food. Suppose now that an average family with an income of $2,000 has its income increased to $2,100. What can the expenditure curve tell us about the change in its food expenditure?

The expenditure curve must be interpreted in this manner:

There are strong presumptions that the family whose income was increased to $2,100 will spend $830, after this family has had sufficient time to become as accustomed to the $2,100 income and the economic situation with it as families now receiving $2,100.

In a period of rapid economic change, as in a business recession, the expenditure curves will be fairly short-run, since incomes will be changing rapidly. Not all of the curve will be of the same length of run, however, for the reason that incomes change more rapidly at some income levels than at others.

In spite of the difficulties of measuring—and interpreting our measurements of—budgetary behavior, analyses of the available data on consumer expenditures will, we feel certain, lead to the discovery of relationships between changes in income and expenditure which are reliable enough to be of great value.

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PERIODICITY AS AN EXPLANATION OF VARIATION IN HOG PRODUCTION

By P. D. Bradley, Jr. and W. L. Crum

This paper[1] presents the results of a statistical analysis of the time series of hog production for the United States. The data forming the series were compiled by the United States Department of Agriculture,[2] and are given in Table 1. The series represents the number (monthly) of federally inspected hogs slaughtered in the United States from 1879 to 1935. With respect to the homogeneity of the hog series it is of interest that the data from 1879 to 1907 represent the equivalent of federally inspected slaughter as estimated by the Division of Crop and Livestock Estimates, while the data from 1907 to 1935 are the actual count as reported by the Bureau of Animal Industry.[3]

Certain general characteristics of the series may be noted briefly. An exceptionally strong and nonvariable seasonal pattern exists throughout. In fact, if the values of all Januaries, Februaries, etc., are successively summed and averaged (see Figure 1, $P=12$) their pattern may be taken to represent, with certain minor exceptions, the movement for any year selected at random. The function describing the trend of the series would show a gradual monotonic rise to about the year 1923, a constant level in the twenties, and a tendency to decline in the thirties.

The analysis reported in the succeeding pages was not preceded by operations of any kind, such as removal of a trend or seasonal, on the original series.

The primary purpose of this study was to determine the existence, if any, of a period[4] or of periods in the hog series. The method used is known as periodogram analysis. Various types of the periodogram have been proposed both before and since Sir Arthur Schuster in a series of articles[5] developed the most widely used and famous method, the Schuster periodogram. The Schuster method has been applied in a

[1] This study was assisted by a grant made jointly by the United States Department of Agriculture and the Harvard University Committee on Research in the Social Sciences. The analysis was carried out by Mr. Bradley with the collaboration of Professor Crum and with helpful advice from Professor John D. Black, Professor E. B. Wilson, and Mr. L. F. Page.

[2] *Livestock, Meats, and Wool,* U.S. Bureau of Agricultural Economics, 1935, p. 69.

[3] *Ibid,* p. 69.

[4] For a discussion of the various meanings of the term, period, see "The Periodogram of American Business Activity," by E. B. Wilson, *Quarterly Journal of Economics,* Vol. 48, 1934, p. 376.

[5] *Terrestrial Magnetism,* Vol. 3, 1898; *Cambridge Philosophical Society Transactions,* Vol. 18, 1899; *Philosophical Transactions, Royal Society of London,* Vol. A 206, 1906.

TABLE 1
HOGS: FEDERALLY INSPECTED SLAUGHTER IN THE UNITED STATES, BY MONTHS, 1879–1935*
(in thousands)
Source: *Livestock, Meats, and Wool,* U.S. Bureau of Agricultural Economics, 1935, p. 69.

Year	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
1879			934	935	1341	1238	739	736	940	1706	2028	1800
1880	1513	1203	1152	1274	1335	1410	1062	1032	1040	1396	2296	1640
1881	1681	1144	827	895	1102	1270	1037	976	1094	1237	1964	1829
1882	1566	951	870	1058	1253	982	767	671	535	776	1680	2049
1883	1694	1042	678	608	935	997	887	804	852	1304	2245	1891
1884	1389	918	614	810	1102	1160	904	883	673	1328	1657	2651
1885	2053	1174	945	1003	1213	1389	1055	811	879	1619	2396	2137
1886	1547	1135	1105	1137	1458	1799	1328	1015	1237	1230	2242	2195
1887	1434	1053	797	835	1231	1318	999	840	912	1197	2970	2390
1888	1724	1492	893	1037	1292	1260	861	670	654	1180	2217	2329
1889	1914	1643	1224	988	1509	1677	1244	824	933	1387	2098	2738
1890	2846	1800	1180	1365	1876	1883	1887	1868	1297	1800	2840	2915
1891	3309	2205	1755	1047	1209	1511	981	767	929	1674	3170	3443
1892	3245	1585	1168	1261	1929	1614	1837	1335	1113	1372	1910	2151
1893	1762	1094	921	1054	1393	1408	1437	1238	1283	1172	1681	1925
1894	2080	1519	1498	1410	1586	2027	1256	1676	1124	1564	2820	2710
1895	2380	1983	1717	1244	1912	1724	952	897	987	1961	2699	2747
1896	2330	1703	1549	1740	2171	2171	1358	1181	1403	2049	1979	2594
1897	2704	2211	1701	1622	2400	2492	1965	1727	1648	1951	2624	3020
1898	2915	2570	2034	2095	2854	2429	2248	1914	1943	2338	2976	4008
1899	3249	2277	2153	1981	2584	2814	2240	1723	1637	2354	2728	2957
1900	3072	2469	2159	2290	2603	2540	2121	1833	1846	2534	2766	3061
1901	3035	2747	1963	2219	2816	2645	2585	2194	1707	2243	3353	3622
1902	3023	2609	2041	1836	2254	2273	1711	1586	1448	2017	2635	2942
1903	2778	2210	1799	1918	2409	2472	2358	1855	1648	1736	2516	3272
1904	3265	2899	1936	2189	2682	2613	1721	1980	1746	2164	3018	3859
1905	3511	2841	2293	2193	2542	3115	2368	1931	1969	2380	3216	3496
1906	3465	2904	2371	2266	2837	3166	2452	2446	1881	2117	2626	3079
1907	3410	2921	2665	2667	3317	3241	2929	2301	1988	2219	2135	3094
1908	4961	3890	3111	2304	3088	3094	2416	2231	2231	3368	3803	4147
1909	3876	2653	3013	2343	2629	2719	2097	1822	1955	2397	2800	3090
1910	2693	2324	1891	1778	2206	2612	1988	1824	1564	1851	2456	2827
1911	2742	2633	2973	2589	3008	3462	2560	2032	2172	2720	3639	3603
1912	4147	3302	2700	2412	2844	2835	2354	1875	1701	2455	3020	3407
1913	3708	2844	2334	2487	3046	3057	2557	2268	2133	2681	3165	3919
1914	3489	2723	2548	2312	2569	2926	2260	1799	1907	2682	3047	4271
1915	4274	3885	3446	2563	2869	3246	2493	2041	1890	2494	3739	5442
1916	5387	4276	3430	2853	3275	3163	2530	2517	2287	3327	4771	5267
1917	4629	3484	2985	2645	3084	2685	2411	1705	1322	2195	3043	3723
1918	3961	3998	3926	3290	3092	2783	2940	2283	1980	3018	4280	5662
1919	5846	4266	3443	3208	3743	3728	2884	1949	1997	2686	3270	4790
1920	5079	3104	3482	2590	3585	3566	2644	2191	1979	2487	3329	3985

* 1879–1906, equivalent of federally inspected slaughter estimated by Division of Crop and Livestock Estimates. 1907–1935, federally inspected slaughter as reported by the Bureau of Animal Industry. Purchases on Government account for the Emergency Hog Production Control Program from August 22 to October 7, 1933 are excluded.

TABLE 1.—*Continued*

Year	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
1921	4347	3799	3047	3003	3274	3618	2821	2530	2422	2866	3447	3807
1922	3985	3480	3350	2946	3716	4046	3104	2888	2747	3332	4318	5201
1923	5134	4231	4838	4179	4325	4303	3983	3556	3212	4328	5341	5904
1924	5911	5006	4536	4073	4278	4288	4114	3070	2857	3498	4641	6600
1925	5979	4447	3299	3037	3186	3732	2819	2453	2598	3314	3646	4533
1926	4501	3351	3562	3105	3131	3430	3127	2834	2616	2976	3610	4394
1927	4514	3395	3837	3330	3766	4253	3431	3050	2534	2969	3688	4869
1928	5479	5780	5140	3446	3884	4078	2984	2545	2508	3713	4455	5782
1929	5738	4478	3645	3761	3798	3756	3597	3130	3104	3857	4499	5083
1930	5001	4034	3392	3480	3823	3689	3187	2724	2773	3492	4024	4647
1931	5362	4142	3523	3488	3408	3251	2767	2500	2955	3772	4218	5387
1932	5027	4590	3664	3714	3940	3320	2802	2970	3252	3605	3778	4584
1933	4700	3647	3602	3847	4286	4626	3914	3477	3038	3058	4501	4530
1934	5391	3433	3039	3411	4218	3763	3324	2641	2601	3545	4312	4197
1935	3048	2409	2158	2178	2172	1828	1712	1668	1453	2135	2422	2875

wide variety of fields where investigators have been interested in the problem of periodicity. Thus, among other fields, it has been used for analysis of rainfall and temperature records,[6] of data on sunspots, and for economic time series.[7] That the results have not been happy and fail to establish any highly probable periods may be ascertained by reference to nearly any of the studies made. The difficulty existed not so much in the method as in the fact that the phenomena studied failed to exhibit periodicity in any strict sense. Nevertheless, chiefly because of the laboriousness of the computations involved in the Schuster method (other difficulties will be mentioned later) various other forms of periodogram analysis have been proposed and applied.[8]

Two of these methods were applied to the hog series, namely, that due to Professor D. Alter and a method developed by E. T. Whittaker and G. Robinson.[9] The first of these methods commences by computing the successive sums between the original series and the original series lagged by a number of items corresponding to the trial period being investigated. Thus, if the original series is represented by $x_1, x_2, \cdots, x_i, \cdots, x_n$, then

[6] D. Brunt, *Geographical Journal*, Vol. 89, No. 3, 1937, p. 214.

[7] E. B. Wilson, *op. cit.*; H. L. Moore, *Economic Cycles, Their Law and Cause*, 1914; and *Generating Economic Cycles*, 1923; W. H. Beveridge, "Wheat Prices and Rainfall in Western Europe," *Journal of the Royal Statistical Society*, Vol. 85, 1922, pp. 412–459; W. L. Crum, "Cycles of Commercial Paper Rates," *Review of Economic Statistics*, Preliminary Vol. 5, 1923; B. Greenstein, "Periodogram Analysis with Special Application to Business Failures in the United States, 1867–1932," *Econometrica*, Vol. 3, No. 2, April, 1935, pp. 170–199.

[8] See K. Stumpff, *Grundlagen und Methoden der Periodenforschung*, 1937.

[9] D. Alter, "An Extremely Simple Method of Periodogram Analysis," *National Academy of Sciences*, Vol. 19, 1933, p. 335; E. T. Whittaker and G. Robinson, *Calculus of Observations*, Ch. XIII, 1924.

$$M_l = \frac{\sum_{i=1}^{n-l} [x_i + x_{i+l}]}{(n - l)}$$

is plotted as ordinate against l as abscissa. The brackets indicate that absolute values are summed. In each case l is a lag corresponding to a trial period. If the data are repetitive after the time l and with the period l, M_l will be large. The best period is that l which gives a maximum M_l. Alter develops other relationships for carrying the analysis beyond this point.[10]

Several computations of M_l were made for l taken between 36 and 55 months and for l between 90 and 100 months.

The maximum value of M_l occurred for $l=48$. The peak was poorly defined with respect to both height and breadth. A more serious objection, however, can be urged against acceptance of 48 months as a probable period in the sense indicated. The movement given by a period of 48 months is merely a repetition of the seasonal pattern. Owing to the analytical construction of the Alter method one would expect that any multiple of the seasonal would appear as a peak in this periodogram if the seasonal were as pronounced and regular as in the hog series. Thus a maximum value for $l=96$ (8 years) was obtained relatively to values of l taken in the immediate vicinity of eight years.

It is not claimed that this was a sufficient test of the effectiveness of the Alter method. Had the seasonal movement been removed from the series better results would perhaps have been obtained. Despite the obvious time-saving advantages of this method, it may be open to question on the ground that it fails to magnify the period sufficiently for economic series. This difficulty might be met by arranging the data, not in two rows $x_{i+l}+x_i$ or cycles, but in as many as the length of the series would permit $x_i+ \cdots +x_{i+re}$. This familiar technique would magnify the trial period under consideration while eliminating others by putting them out of phase.

The correlation periodogram consists in arranging the items of a series $(x_1, x_2, \cdots, x_n)$ in successive rows and columns until the series is exhausted. The length of the row corresponds to the trial period to be investigated; variations in the length of the row correspond to changes in the trial period. If there are K items in the series, then for a trial period equal to P there will be M items in each column ($PM=K$). The number of rows will indicate the total number of repetitions of the period in the data and the columns will indicate the phase of the period. Each column is summed and averaged (designate the column sums by $X_1, X_2, \cdots, X_P$ and the averages by $\overline{X}_1, \cdots, \overline{X}_P$), and the

[10] *Ibid.*, p. 336.

correlation ratio $\eta_P = \sigma^2_{\bar{x}_i} / \sigma^2_{X_P}$ is computed. The ratio η_P is then plotted as ordinate against P as abscissa, and the best P is that which has the maximum η_P.

The results obtained by this method were decidedly inconclusive. The best periods were merely multiples of the seasonal movement; thus for $P=12$, 24, 36, and 48 months, η equals 0.49 approximately. The next best values obtained were for $P=30$ months, $\eta=0.338$; and for $P=42$, $\eta=0.335$. All other values for η were less than 0.14. (For

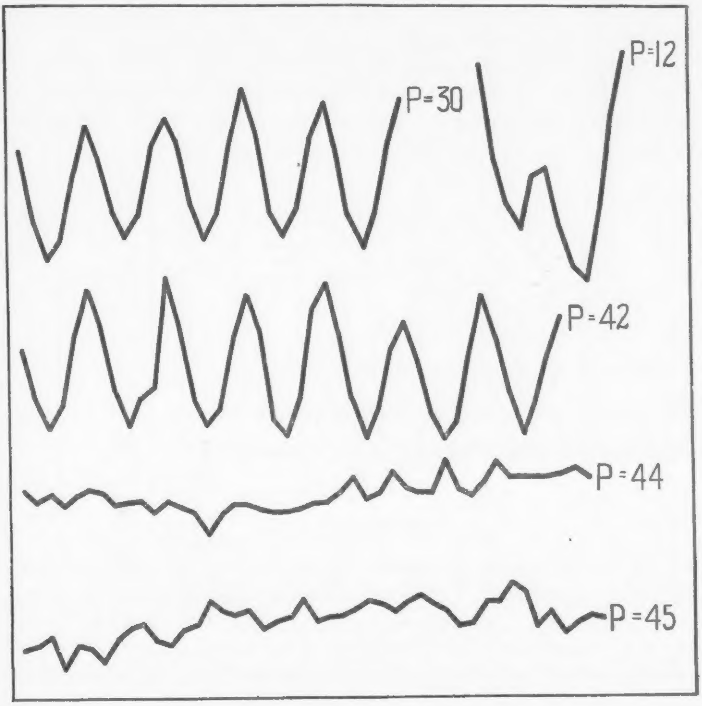

FIGURE 1.—Comparative movements of the column averages for periods of 12, 30, 42, 44, and 45 months.

$50 > P_i > 9$.) The column averages for $P=30$ and $P=42$ have been plotted in Figure 1, $P=30$, $P=42$. The form of the movement strongly suggests that of the seasonal (Figure 1, $P=12$). It is evident that 30 months is merely two and one-half seasonals while 42 months is three and one-half. If for no other reason, therefore, we may abandon the 30-

and 42-month periods. Figure 1, $P=42$, can be described by an expression of the form

$$x = C + A \cos pt + B \sin pt$$

(where $p=2\pi/P$). The deviation about the curve, however, would be so great as to annihilate any possibility of establishing a high degree of correlation between the data and the sinusoidal.

Up to this point no periods of significance were uncovered. Consequently the laborious computations of the Schuster method were undertaken not only to further the search for possible periods, but also to compare the relative usefulness of the three methods.

Schuster's method requires that the data given by the series be formed into rows and columns as in the case of the correlation periodogram. Again designating the column sums for any given trial period by $X_1, X_2, \cdots, X_P$, the following expressions are computed:[11]

$$A = \frac{2}{k}\sum_{i=1}^{P} X_i \cos pt, \tag{1}$$

$$B = \frac{2}{k}\sum_{i=1}^{P} X_i \sin pt, \tag{2}$$

$$S = A^2 + B^2. \tag{3}$$

A and B are the amplitudes of the equation

$$x = C + A \cos pt + B \sin pt, \tag{4}$$

which is the fundamental term of the Fourier series

$$x = C + A \cos pt + \cdots$$
$$+ B \sin pt + \cdots .$$

K is the total number of items in the time series and $p=2\pi/P$. S is called the periodogram coefficient. Plotted against P it may be taken as a measure of the intensity of the period. As in the other two methods, S is a maximum when P is chosen in the neighborhood of the true period or periods in the data. It is seen from the form of the expressions (1), (2), (3) that the value of P is sought which will give the largest possible amplitudes existent in the series. The best period in this sense is then the one for which the amplitudes are maxima. Before considering other aspects of the Schuster method the results of the computations of S will be given.

The investigation centered chiefly on trial periods between nine and fifty-four months where values of P were taken one month apart, al-

[11] E. B. Wilson, *op. cit.*; D. Brunt, *Combination of Observations*, Ch. XIII.

though some few periods were tried beyond fifty-four months. In the hog series ($K=672$) this meant that a period of nine months would repeat itself some seventy-five times whereas a period of fifty-four would be manifested only twelve times.[12] The periodogram for the range between nine and fifty-four months is plotted in Figure 2.

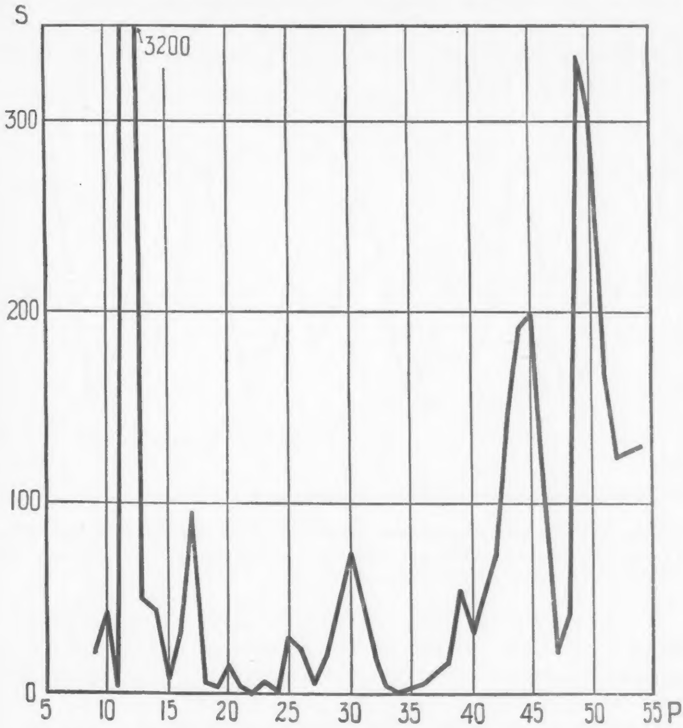

FIGURE 2.—The Schuster periodogram of the time series of hog production in the United States (1879–1935).

The most conspicuous peak is for $P=12$. This was to be expected from the intensity and regularity of the seasonal pattern. Evidently the seasonal configuration would account for a large part of the variation of the series.

[12] Nearly all periodograms computed for economic data suffer on this account. Moore's 8-year cycle was repeated five times while his 33-year cycle showed itself but once in his series covering forty years—1870–1910. Greenstein's period of 9.4 years would be contained nearly seven times in his record of sixty-six years.

If the period equal to twelve months is excluded as scarcely of the type being sought, the best period is then evidently in the neighborhood of 49 months. The form of the movement for $P=48$ through 52 months can be seen in the column averages ($\overline{X}_i$) plotted in Figures 3 and 4. The peak at 49 months is well defined with respect to intensity and breadth. There is clearly no period, in the sense defined, for 48 months. The relatively small value for S when $P=48$ months may be explained by the fact that equation (4) does not describe the movement delineated in Figure 3.

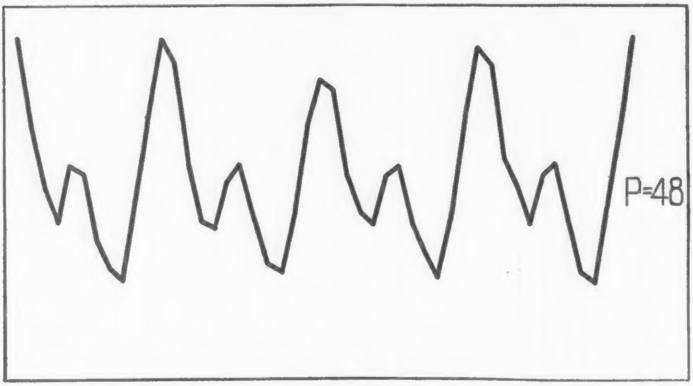

FIGURE 3.—Movement of the column-averages for a period of 48 months. This movement is almost exactly equivalent to the one obtained by repeating the movement for $P=12$ four times (see Figure 1).

For $P=49$, $S=329$. The expected value of S over the range computed is 62; hence, the value for S corresponding to a trial period P of 49 months is some 5 times the expected value. For $P=50$, $S=302$, S is again nearly 5 times the expected value. Further grounds for belief that a true period exists in the neighborhood of 49 or 50 months is given by Figure 4. Both $P=50$ and $P=49$ of Figure 4 could be described by equation (4) and especially $P=49$.

The periodogram establishes a period of 49 months as the most conspicuous over the range considered. The method does not eliminate the possibility that the form of the movement was the result of the additive composition of irregular movements, that the amplitude is variable, that the movement is nonsymmetrical, or that the period may not hold from one section of the data to the next.

The following problem might be considered: for what value of K is S a maximum? In other words, holding P constant, vary K, and find that

value of S which is a maximum. Table 2 presents the results of this experiment.

K was varied by computing S for the last six hypothetical cycles in the time series, then by adding two more cycles and again computing

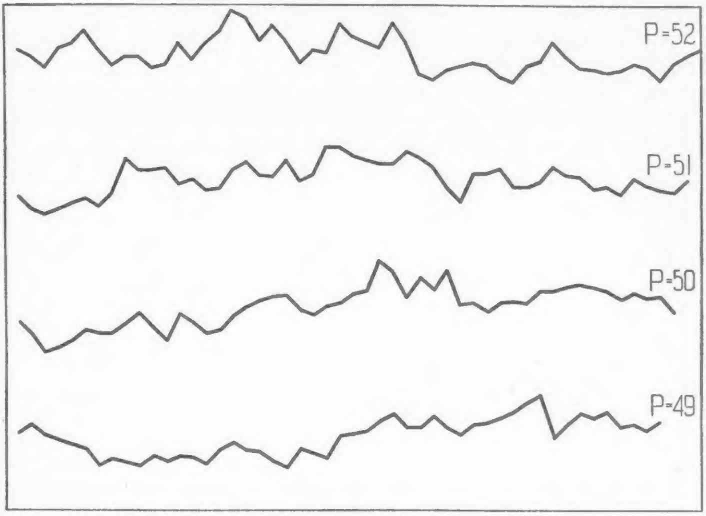

FIGURE 4.—Movement of the column averages for periods of 49, 50, 51, and 52.

S, etc. Finally, for $K=637$, or for 13 cycles, the historic time interval included was from January, 1933 to January, 1880. It would seem from the evidence of Table 2 that the amplitude becomes more pronounced as the movement away from the origin (1880) increases. Or, considering 1933 as the origin, the amplitude is damped as more cycles are in-

TABLE 2

P	K	M	S	A	B	$\omega°$
49	294	6	947	+9.1	−29.4	163
49	392	8	783	+7.7	−26.9	164
49	490	10	524	+5.6	−22.2	166
49	637	13	329	+2.6	−17.9	172

cluded in K. The period holds better for the later years of the time series than for the earlier years. Further light is thrown on the nature of the period involved by a consideration of the plots of the column averages ($\overline{X}_i$) for the various values of K as shown in Figure 4, $P=49$,

and Figure 5, *a*, *b*, *c*. The regularity of the curve in Figure 4 would seem to be the result of the combination of the irregular curves in Figure 5. Any argument as to the reality of a period on the basis of large values of S would be deceptive in this case. The curves of Figure 5 could scarcely be said to have the form

$$x = C + 2.6 \cos \frac{2\pi}{49} t - 17.9 \sin \frac{2\pi}{49} t,$$

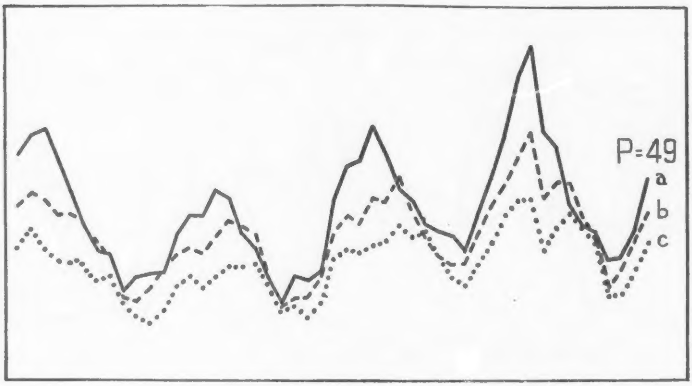

FIGURE 5.—Movement of the column averages for a period of 49 months when different values are assigned to K. The last six cycles in the time series are averaged and plotted in *a*. Two more cycles are added to obtain *b*, and another two cycles are added to obtain *c*.

especially for the first six and eight cycles. No particular sinusoidal movement is magnified by the arrangement of the data into rows and columns but rather, it seems, diverse movements are averaged out.

It is customary in periodogram analysis to split the series into independent sections and compute S for each. This was done for the two halves of the series and the results are given below (Table 3).

TABLE 3

	P	K	M	A	B	S	$\omega°$
1st half	49	343	7	−2.8	−8.2	75	199
2nd half	49	294	6	+9.1	−29.4	947	163

The conclusions drawn from the data of Table 2 are substantiated by the data of Table 3, as should be expected from the nature of the computations. While the form of the movement for 13 cycles is definitely

wavelike (Figure 4, $P=49$), the movement for the first half of the series (7 cycles) is definitely not, at least it has not the form of equation (4) which is the type of equation involved (see Figure 6). Likewise the movement of the first half differs from that of the last half.[13]

An important test for the reality of a period is the degree of variation in the phase angle, ω. The expression $\tan \omega = B/A$ yields the maximum phase of a sine curve fitted to the period of the data. It does not give the maximum phase of the period in the data itself, unless corrected. If the latter period should be symmetrical, i.e., if it is like a sine curve, then its maximum phase will coincide with that of the former. From Table 2 it is clear that the variation in the phase angle is not great. For six cycles the maximum phase occurs at 163° or (with the

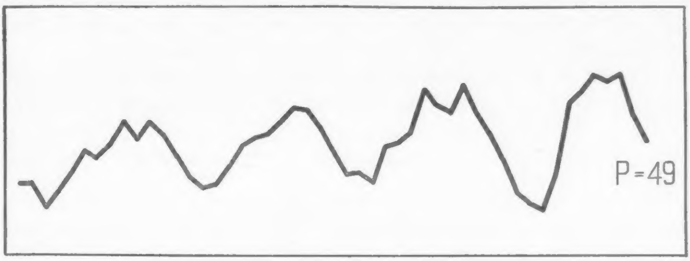

FIGURE 6.—Movement of the column averages for the first 7 cycles of the hog series (1880–1908) for a period of 49 months.

limitation suggested above) at twenty-two months. For thirteen cycles the maximum occurs at 172° or at 23 months. It will be recalled that the data of Table 2 were computed by setting $P=49$ and varying K, and the value of ω obtained seem to substantiate the conclusion that while the length of the period and its form are substantially constant (although not symmetrical), its intensity is increased with the distance from the origin (1880). The variation in the phase angle is increased when the series is split into two sections (Table 3). For the first half, the maximum phase occurs at 27 months while for the last half it occurs at 22 months. It seems to the present writers that inspection of the plot of column averages is of greater assistance in this connection than the computations of the phase angle.

A further test for the reality of a period suggested by the periodogram arises out of the nature of the mathematical method used. If a true period lies between two trial periods, P_i and P_{i+1}, then, unless

[13] It is of interest that the first half (7 cycles) covers the interval from 1880 to 1908 (approx.). In 1907 a different method was used for collecting the hog data (see p. 221 above).

there is interference by another period, the amplitudes A and B will change signs between trial periods. Such a change of signs occurs between $P=47$ and $P=54$. There is no such change between $P=48$ and $P=50$.

TABLE 4

P	42	43	44	45	46	47	48	49	50	51	52	54
A	−1.7	+11.3	+6.3	−9.9	−9.7	−2.9	+6.3	+2.6	−10.4	−13.0	−4.9	+.49
B	8.3	−4.2	−12.3	−10	−2.9	+3.8	−1.2	−17.9	−13.9	−0.14	+9.9	−10.9
S	71.7	145	191	198	103	23	41	329	302	169	123	119

In general this test must be inconclusive because for any actual series the signs may change without the presence of a true period. Likewise, interference may exist, thus preventing a change in sign although a true period is present. Accepting the test, the most that can be said is that the period lies between 47 and 54 months.

It might be well to consider the possibilities of interference between the peak at 49 months and the next most prominent one at 45 months. In works on the theory of the periodogram the following inequality is developed as a test for absence of interference between two real periods:

$$Q_2 - Q_1 > \frac{2Q_1Q_2}{K} \cdot$$

The Q_1, Q_2 are two periods of a sine curve and to guard against interference K must be so large that the inequality holds. Substitution of $P=45$ and $P=49$ reveals that the length of the series is not sufficient to insure noninterference between the two peaks. In fact, given $K=672$ the two peaks would have to occur at $P=45$ and $P=52$ in order to eliminate the possibility of interference.

A complete analysis of the properties of the peak at 45 months was not carried out. For $P=45$, S equals 198, which is some 3.2 times the expected value of S over the range computed. Figure 1 ($P=44$, $P=45$) presents the plot of the column averages for 44 and 45 months. The form of the movement is of the type required by the periodogram although not so distinctly as for the 49- and 50-month periods.

If all of the periods indicated by the periodogram are accepted and considered as reflections in the series of certain real forces, physical and economic, do they provide sufficient grounds for explaining the variation in the series? The total variation of the series is described by the standard deviation, which for the hog series is $\sigma=114$. The formula $\sigma^2-\frac{1}{2}\sum S_i=V$ provides a basis for an answer to the question. The S_i are those ordinates of the periodogram which are of such intensity as to indicate the approach of a trial period to a real period, and V is the amount of the variation unaccounted for by the periods selected on

the basis of the periodogram. The S_i corresponding to $P=12$, 45, and 49 were chosen as the most prominent. It was found that some 15 per cent of the variation of the series was accounted for by these three periods. The question must be answered in the negative if the magnitude of σ is considered as the result of certain periodic forces. That is, if an expression of the form

$$
\begin{aligned}
x = 260 &+ 57 \cos \frac{2\pi}{12} t - 0.72 \sin \frac{2\pi}{12} t \\
&- 10 \cos \frac{2\pi}{45} t - 10 \sin \frac{2\pi}{45} t \\
&+ 2.6 \cos \frac{2\pi}{49} t - 17.9 \sin \frac{2\pi}{49} t
\end{aligned}
\tag{5}
$$

were constructed, the probability that values of x as given by (5) would be a good approximation to the actual values, as given by the series, would be very small.

CONCLUSIONS

The strong and persistent seasonal running through the hog series has made hazardous the application of all three periodogram methods. This is especially true in the case of the Alter and the Whittaker and Robinson methods where the best periods are merely multiples of the seasonal. The difficulty is partially avoided in the Schuster method because of the requirement that the column averages have the form of equation (4). On the other hand, this last requirement introduced difficulties of its own. It definitely limits the type of period to sine curves (one maximum and minimum) of large amplitude; repetitions of movements other than of sine form would not be indicated in the periodogram. Furthermore, the use of smoothing operations tends to introduce spurious periods. Professor Alter has indicated that the effects of smoothing can be allowed for in his periodogram, although time did not permit a trial of this in the present study.

The Schuster periodogram indicates a fairly strong period in the neighborhood of 49 or 50 months. The movement is not symmetrical, nor is it persistent throughout the time interval of the series. The change in intensity of the 49-month period coincides roughly with a change in the method of collecting the data. From 1907 to 1933 the intensity of the period is tremendously increased. Unfortunately the length of the series permits only six manifestations of the period after 1907, and this fact makes all conclusions extremely tentative.

What is the relationship of such studies as this to business-cycle theory? Many pages have been written on the subject of business

cycles despite a general inability to define the term itself. Some writers have resorted to the cyclical behavior of individual series. Theories have been erected on the supposition that they would be substantiated if the data did exist. There are, however, certain difficulties in this approach. What is meant by *cycle*? Certainly *cycle* is intended to connote something less strict than the term period. A very loose definition would define the length of the cycle as $P \pm x$, where x might vary and yet be fairly small relatively to P; likewise the amplitude and phase might be variable. It is not clear how a looser definition of the term could be made and still have meaning. The evidence produced tending to prove the existence of cycles even in this sense is extremely unsatisfactory. A presumption towards belief in a cycle must be created by the empirical evidence. Let us suppose that a time series of yearly data covers the eighty-year interval from 1850 to 1930. This would mean that the famous eight-year cycle would have only ten opportunities for repetition while the thirty- to forty-year cycles could repeat themselves but twice. Inasmuch as these cycles may admittedly vary in length the adequacy of the evidence becomes even more tenuous. Furthermore, economic records of such length are almost certain to lack homogeneity—hogs in 1850 are for all practical purposes of economic analysis not hogs in 1930; yet a shorter series would be entirely unsatisfactory. In the hog series the record would seem to be homogeneous from 1907 on. This would allow some three repetitions of an eight-year cycle. The case becomes, if anything, more difficult for monthly series. Not only do they fail to provide sufficient evidence of cycles, but nearly all monthly economic series are affected by strong seasonals. Cycles in the neighborhood of years, or even half years, may possibly be merely multiples of the seasonal movement. Removal of the seasonal tends to introduce spurious cycles. Hence, attempts to establish empirical evidence in favor of certain cycles are beset by two difficulties which result in an impasse: (1) the series are too short, or, being long, they are nonhomogeneous; (2) cycles in the neighborhood of any given number of years are likely to be the result of the seasonal, while removal of the seasonal creates spurious cycles.

The writers are aware that these objections are contained in many footnotes to business-cycle studies; but they have been passed over as annoying and trifling difficulties which can and will be surmounted when economic series are improved. Few have questioned the adequacy of the notion for scientific discussion. It might be well, however, to reconsider the hypothesis that economic nature expresses itself in cyclic form.

Harvard University

THE PRACTICE OF DEPRECIATION

By Gabriel A. D. Preinreich

In a previous article[1] I made a brief and incomplete survey of the theory of depreciation. In the present paper I discuss its practice. One obstacle to practical progress in this field is that mathematically trained minds are seldom well informed on what accountants actually do. The latter are therefore more often criticized for methods which they are not using than for those which they are. Even otherwise valuable contributions thus elicit opposition quite unnecessarily. The inappropriate antithesis tends to discredit the rest of the argument and prompts general retorts, for instance that "accounting is a tool of business . . . determined by the practices of business men.—Where accounting treatment diverges from economic theory, a similar divergence is likely to be found between economic theory and business practice."[2] Such an attitude, in turn, is not very helpful or progressive, even if the dangerous phrase "tool of business" is interpreted only in its best possible sense.

In the article cited, I probably added to the already existing confusion by calling sample methods by certain names without proper qualification, although the same names are commonly applied to substantially different methods. The truth is that the familiar "single-machine" formulae permit of different interpretations. To clarify the situation, the present paper identifies a greater number of methods unequivocally by developing their basic "many-machine" equations and comparing the results. References to practice and to individual writers' ideas are made wherever possible, before choosing a method which appears best suited to the practical needs of large enterprises and the investing public.

I

The task of translating the net effect of many debits and credits into a single mathematical formula requires some familiarity with certain elementary concepts used by statisticians. Although these were given in my survey of the theory, upon which this paper leans heavily, a restatement of the basic formulae seems desirable.

When a large number of similar machines is installed at the same time, the rate at which they will drop out of service forms a bell-shaped, but usually skew, *frequency distribution*. Let it be drawn in such a form

[1] "Annual Survey of Economic Theory: The Theory of Depreciation," Econometrica, Vol. 6, July, 1938, pp. 219–241.

[2] George Oliver May (Chairman, Committee on Accounting Procedure, American Institute of Accountants), "The Influence of Accounting on the Development of an Economy," *Journal of Accountancy*, January, 1936, pp. 11–12.

that its ordinates $f(0)$ at the time $t=0$ and $f(n)$ at the time $t=n$ be zero and that the total area enclosed by the curve and the axis of abscissae be equal to unity. For any other time between these limits the curve will be denoted by $f(t)$. Summation of $f(t)$ from t to n gives the *mortality curve* $M(t)$ per centum of the total number of machines installed at the time $t=0$:

$$(1) \qquad M(t) = \int_t^n f(\tau)d\tau; \qquad M(0) = 1; \qquad M(n) = 0.$$

It will sometimes be necessary to employ the mortality or survival function in the inverted form $M^{-1}(y)$, which expresses the abscissa corresponding to any given ordinate y. The usual form $M(t)$ states the ordinate corresponding to any given abscissa t.

The area enclosed by the mortality curve and the co-ordinate axes represents the total life units of service expected from the machines installed at the outset. Since the ordinates are expressed per centum, the area also equals the *average life* of the machines, i.e., their life expectancy at $t=0$. The *life expectancy* of the machines still in service at any other time t is the quotient of the remaining life units of service by the mortality curve:

$$(2) \qquad L(0) = a = \int_0^n M(\tau)d\tau; \qquad L(t) = \frac{1}{M(t)}\int_t^n M(\tau)d\tau.$$

In accord with this introduction, all formulae given hereafter take it for granted that the number of machines composing the plant is large enough to permit the use of continuous functions. Attention necessarily centers on the *undepreciated remainder per centum of the original wearing value* (i.e., cost less present worth of the scrap value) *of a large number of machines installed together*. This concept is denoted throughout the present paper by $r(t)$. It varies between $r(0)=1$ and $r(n)=0$ and will be defined in many different ways by as many different depreciation methods numbered consecutively. The *rate of depreciation* is $dr(t)/dt$ or $r'(t)$ for short. Scrap values are omitted, because it is simpler to consider them separately. Any remainder of wearing value $r(t)$ can be readily converted into the *book value or unexpired cost per centum* $c(t)$ by the operation:

$$(3) \qquad c(t) = \left[1 - s\int_0^n f(\tau)e^{-i\tau}d\tau\right]r(t) + s\int_t^n f(\tau)e^{i(t-\tau)}d\tau.$$

where s=*scrap value per centum* of original cost. When the rate of interest is zero, this formula reduces to:

$$(4) \qquad c(t) = (1 - s)r(t) + sM(t).$$

The *depreciation rate on the book value* is $c'(t)+sf(t)$ in either event. In the survey of theory book values were used instead of wearing values. The notations are uniform in both papers, but a glossary of symbols is also submitted in the present Appendix B.

The presentation is entirely general, i.e., valid no matter what the shape of the mortality curve may be. Only for the sake of *graphic illustration* did I adopt the concrete sample:

$$\textbf{(5)} \qquad M(t) = \frac{12}{n^4}\int_t^n \tau(n-\tau)^2 d\tau; \qquad n = 20 \text{ years}.$$

My assumption of the maximal age n is purely arbitrary; in other respects, however, the function happens to reflect fairly closely the behavior of seventeen different types of equipment out of a total of fifty-two studied by Prof. Edwin B. Kurtz.[3]

Equation numbers appearing in heavy type in the text mean that those equations are plotted in the form of curves similarly numbered in one of the four figures submitted. For any formula containing either $M(t)$, or its derivative $-f(t)$, the figure is obviously valid only when these symbols refer to the sample **(5)** above.

To clarify the processes which lead to the continuous functions when the number of machines is very large, all four figures also show what would happen, if there were only ten machines. Each of the ten layers represents a machine, the horizontal dimension being its life graduated according to the sample **(5)** and the vertical dimension its original wearing value per centum of all such values. The area must then express the number of life-service units originally purchased. The method of depreciation is indicated by shading in such a manner that the shaded portions of any ordinate add up to that fraction of the original wearing value of ten machines, which remains unrecovered at any time t under the method illustrated.

The answers obtainable from the shading do not agree exactly with the continuous curves plotted, because a division of the plant into only ten machines is not sufficient. The discrepancies are surprisingly small, however. This shows that any reasonably large number of machines may be considered infinitely large for theoretical purposes. The use of continuous functions is therefore fully justified by the great saving in effort which they permit.

Nearly a score of different depreciation methods will now be analyzed in turn.

[3] *Life Expectancy of Physical Property*, Ronald Press Co., New York, 1930.

II

1. *The Economist's Straight-Line Method—Figure* 1

The vertical line of average life is drawn and the cost of each machine

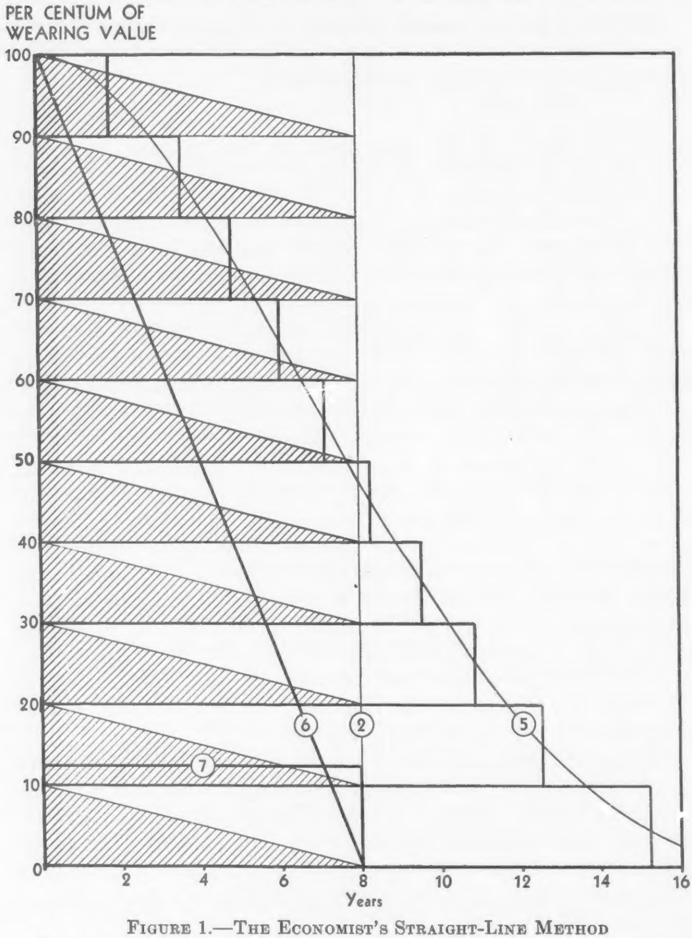

FIGURE 1.—THE ECONOMIST'S STRAIGHT-LINE METHOD

Shaded portions of any ordinate add up to undepreciated remainder wearing value per centum, when plant consists of ten machines of similar type, installed simultaneously. Life characteristics differ according to mortality curve (5).

When number of machines is very large, (6) = remainder wearing value per centum, (7) = rate of depreciation, and (2) = average life.

Maximal life $n = 20$ years. Numbering corresponds to equation numbers of text.

is distributed evenly over that period. The unexpired cost accordingly declines in a straight line, but the rate of depreciation is straight only with reference to the original number of machines, not per machine in service:

$$r_1(t) = 1 - t/a, \qquad 0 \leqq t \leqq a, \tag{6}$$

$$r_1'(t) = -1/a. \tag{7}$$

In terms of debit and credit the method implies the charging of all costs of acquisition to a "plant account." Periodic charges to operations at the rate $1/a$ per annum are credited to a "reserve for depreciation." Whether or not costs are canceled against the reserve a years after purchase is immaterial; the difference of the two accounts always gives the unrecovered cost.

Economists, statisticians, engineers, etc. should note that accountants do not use this method. Accounting textbooks or monographs written by accountants do not even mention it. Outside accounting literature it is nevertheless the standard antithesis which readily demonstrates the superiority of whatever other method a given writer may advocate.

2. *The Accountant's Straight-Line Method—Figure* 2

The plant account is charged with original cost and the reserve credited with depreciation charged to operations at the annual rate $1/a$, but only for a years or the actual life of the machine, *whichever be less*. Upon discarding, a bookkeeping entry is immediately made in the following form:

		Dr.	Cr.
	Reserve for Depreciation	$\Delta yt/a$	
(8)	Loss on Capital Assets	$\Delta y(1 - t/a)$	
	Plant Account		Δy

In this entry Δy represents the original cost of a machine in the absence of a scrap value, as assumed.[4] If the machine outlives the average age a, there will be no loss and the entry made upon discarding is simply:

		Dr.	Cr.
(9)	Reserve for Depreciation	Δy	
	Plant Account		Δy

[4] When there is a scrap value, the credit remains Δy, but the debits shown will refer only to $(1-s)\Delta y$. An additional debit $s\Delta y$ is then necessary to an account which may be called "scrap inventory" for present purposes.

The important point to note is that if the cost of the originally installed machines had been charged to one account and that of their

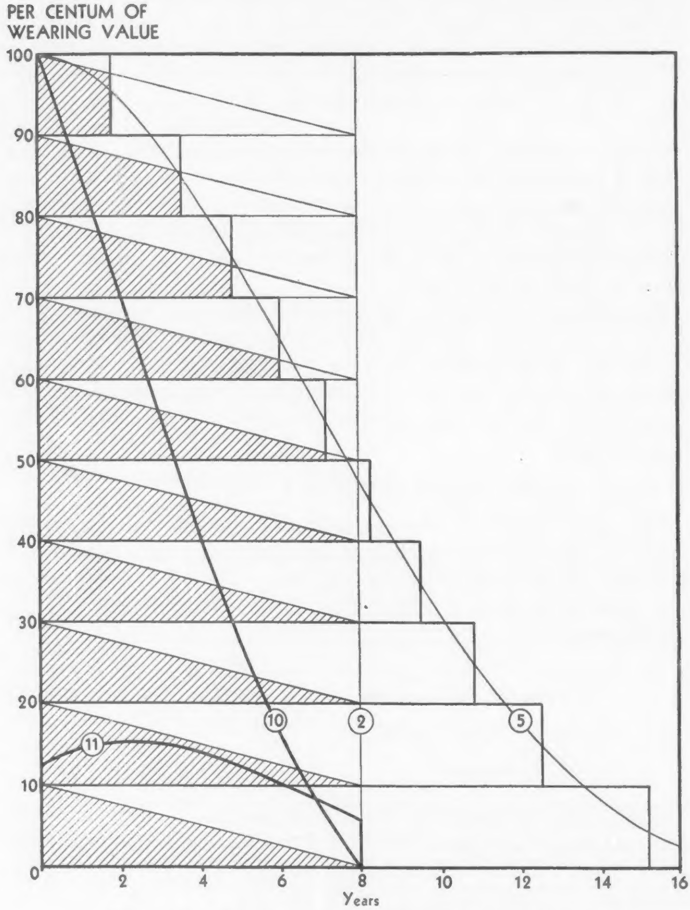

FIGURE 2.—THE ACCOUNTANT'S STRAIGHT-LINE METHOD

Shaded portions of any ordinate add up to undepreciated remainder wearing value per centum, when plant consists of ten machines of similar type, installed simultaneously. Life characteristics differ according to mortality curve (5).

When number of machines is very large, (10) = remainder wearing value per centum, (11) = rate of depreciation, and (2) = average life.

Maximal life $n = 20$ years. Numbering corresponds to equation numbers of text.

replacements to another, the successive balances of the former account would outline the mortality curve. Depreciation is charged at the rate $1/a$ only on the machines in service, but the early capital losses are in effect additional depreciation charges; the method is therefore not straight. Its formulae are:

$$r_2(t) = \left(1 - \frac{t}{a}\right)M(t), \qquad 0 \leqq t \leqq a; \tag{10}$$

$$r_2'(t) = -\frac{1}{a}M(t) - \left(1 - \frac{t}{a}\right)f(t). \tag{11}$$

3. *The True Straight-Line Method—Figure* 3

For practical reasons, method 2 distorts the general principle accepted by all accountants that "the capital sum to be recovered shall be charged off over the useful life of the property. The deduction . . . shall be limited to such ratable amount as may reasonably be considered necessary to recover during the remaining useful life of the property the unrecovered cost."[5] "If it develops that the useful life of the property will be longer or shorter than . . . originally estimated under all the then known facts the portion of the cost . . . not already provided for . . . should be spread over the remaining useful life . . . as re-estimated in the light of the subsequent facts."[6]

When the term "property" is interpreted to mean a large number of machines taken together, the rule evidently suggests that the rate of depreciation allowable is the quotient of the wearing value by the average life expectancy of the machines still in service:

$$r_3'(t) = -\frac{r_3(t)}{\dfrac{1}{M(t)}\displaystyle\int_t^n M(\tau)d\tau} = -\frac{1}{a}M(t), \qquad 0 \leqq t \leqq n; \tag{12}$$

$$r_3(t) = \frac{1}{a}\int_t^n M(\tau)d\tau = \int_0^{M(t)} \frac{M^{-1}(y) - t}{a}dy. \tag{13}$$

Application of the method consists of charging depreciation at the annual rate $1/a$ only on the machines in service, but throughout their entire life, whether that be more or less than a years. No early losses are recognized and the bookkeeping entry is always (9), regardless of when a machine is scrapped. Part of the cost of machines discarded before the age a is thus left, not in the plant account, but in the difference between it and the reserve, i.e., in the book value. This excess is

[5] *Revenue Act of 1936, Regulations 94,* Art. 23(1)–5.
[6] *Revenue Act of 1928, Regulations 74,* Art. 205.

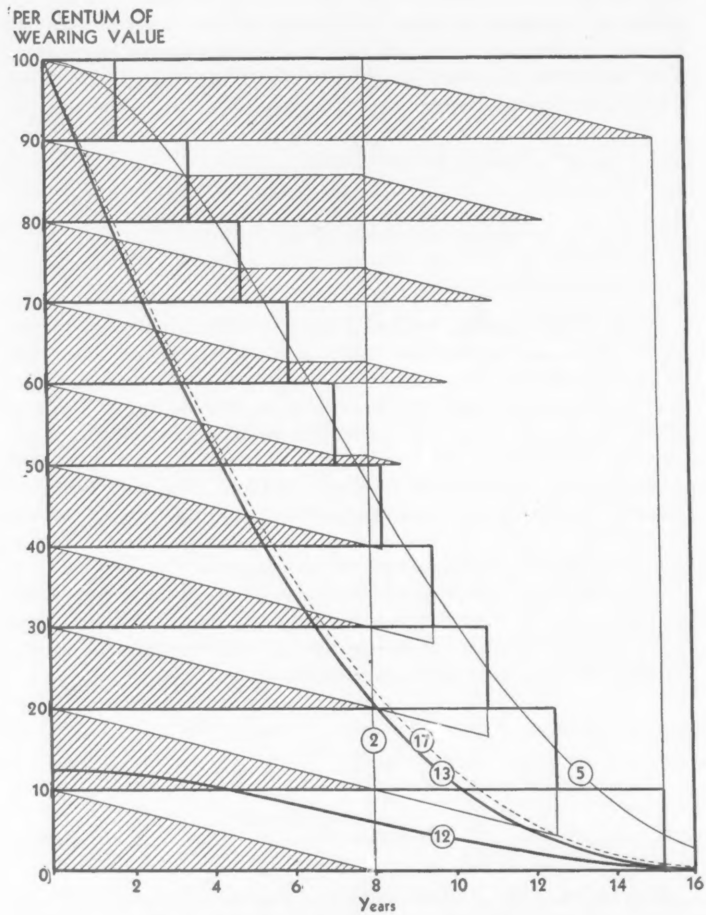

FIGURE 3.—THE TRUE STRAIGHT-LINE METHOD

Shaded portions of any ordinate add up to undepreciated remainder wearing value per centum, when plant consists of ten machines of similar type, installed simultaneously. Life characteristics differ according to mortality curve (5).

When number of machines is very large, (13) = remainder wearing value per centum, (12) = rate of depreciation, and (2) = average life. The broken line (17) = remainder wearing value per centum for Professor Kurtz' method.

Maximal life $n = 20$ years. Numbering corresponds to equation numbers of text.

gradually absorbed by continuing charges at the rate $1/a$ on the balance of the plant account even after machines recorded in it are past the average age. If a was correctly forecast, both the plant account and the reserve will be closed out after the last machine is discarded.

The depreciation rate is here a constant or "straight" rate per machine; the method considers a life unit equivalent to a cost unit, whether that unit be furnished by a short- or long-lived machine. This approach has the merits of *the averaging or insurance principle* so long as early "losses" are not recognized.[7] The principal difficulty is that of estimating the average life correctly. Errors have a cumulative effect on the book value of a composite plant; a separate statistical department is accordingly necessary. Only large enterprises, such as the Bell Telephone System, can afford to use this method.[8]

4. *The Method of Weighted Life Units—Figure* 4

The accounting principle quoted may also be applied to each machine separately. The purchase price of each having been the same, the cost of a life unit is inversely related to the number of units acquired. The weighting is done graphically by drawing a diagonal line across each rectangular layer, or mathematically by the formulae:

$$r_4(t) = \int_0^{M(t)} \frac{M^{-1}(y) - t}{M^{-1}(y)}\, dy = \int_t^n (1 - t/\tau) f(\tau) d\tau, \qquad 0 \leqq t \leqq n; \tag{14}$$

$$r_4'(t) = -\int_t^n \frac{f(\tau)}{\tau}\, d\tau. \tag{15}$$

The rate of depreciation being different for each machine, it is improper to refer to the unrecovered cost as the "remainder service life" per centum of the total useful lives of a large number of machines installed at the same time. That information can be obtained only by omitting the weighting process, i.e., in the manner shown in Figure 3.[9]

[7] The Revenue Act of 1938 again permits the deduction in full of losses as outlined in journal entry (8). In a few prior revenue acts the deduction is limited by the capital gain and loss provisions, but the new definition of capital assets "does not include . . . property used in the trade or business, of a character which is subject to the allowance for depreciation provided in section 23(1)" [Cf. Section 117(a)(1)].

[8] Cf. Allan B. Crunden and Donald R. Belcher, "The Straight-Line Accounting Practice of Telephone Companies in the U. S.," *Proceedings of the International Congress of Accounting*, New York, 1929, pp. 351–386.

[9] The value of Professor Edwin B. Kurtz's book, *The Science of Valuation and Depreciation*, Ronald Press Co., New York, 1937, is seriously impaired by his belief that methods 3 and 4 are equivalent. His Table VIII calculates the cor-

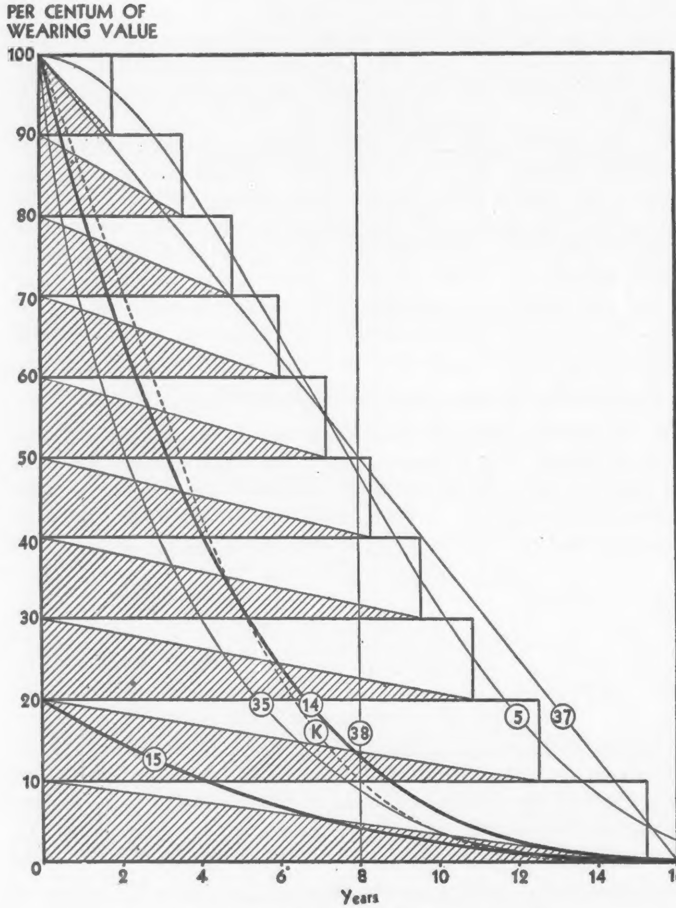

FIGURE 4.—THE METHOD OF WEIGHTED LIFE UNITS

Shaded portions of any ordinate add up to undepreciated remainder wearing value per centum, when plant consists of ten machines of similar type, installed simultaneously. Life characteristics differ according to mortality curve (5).

When number of machines is very large, (14) = remainder wearing value per centum and (15) = rate of depreciation. The broken line (K) = remainder wearing value obtained by Professor Kurtz (see note 36); (35) = example of remainder wearing value for the theoretical public-utility method, when mortality curve is not (5), but coincides with the limit of shape (37). Other limit of shape of all possible mortality curves normalized to same area is (38) = (2) = average life.

Maximal life $n = 20$ years. Numbering corresponds to equation numbers of text.

It may be seen now that method 2 really attempts to follow method 4. Errors in past depreciation rates can be definitely ascertained only upon scrapping. The correction is then made by entry (8), if the machine drops out sooner than expected. In the opposite event the cost has already been written off so that, broadly speaking, it is recovered within the life of the machine in either case. That is about all that can be done at present in general practice. The Bureau of Internal Revenue does not ordinarily disallow the use of standardized rates on the basis of method 2, even though tax blanks demand that new life estimates be made from year to year. Voluntary revisions are made occasionally.

5. *The Kurtz Method of Replacement Insurance*

According to Professor Edwin B. Kurtz this method is "the only scientific approach to the problem"[10] of depreciation. It consists of applying the fundamental insurance formula,[11] which prescribes that the discounted sum of all premiums must equal the discounted sum of all benefits paid. In order to obtain an algebraic approximation, the number of premiums due is measured by the survivors at the beginning of each year and the benefits due by the sudden drop at the end of the year.

In terms of continuous functions the correct premium is evidently:

$$P = \frac{\int_0^n f(\tau)e^{-i\tau}d\tau}{\int_0^n M(\tau)e^{-i\tau}d\tau}. \tag{16}$$

When the discount rate is zero, it follows from this formula that the premium is $1/a$ per machine, i.e., just what it is in method 3. Instead of that, however, Professor Kurtz gets $P=1/(a+\frac{1}{2})$ from his approximation. He explains that "this of course, is not due to any inaccuracy . . . but rather to the difference between the number of installments to be paid. In the straight-line plan the number of units is fixed by the average life, namely 1000, 2000, 3000, or 4000 for the 100 units. In the replacement-insurance plan the number of units in service at the beginning of each age fixes the number of premiums to be collected to make possible paying the 100 benefits. These totals

rect remainder service life by arithmetical processes corresponding to formula 13, but his Figure 16 is similar to the present Figure 4. The equivalence of the two is then "proved" by forcing a sample of Table VIII into Figure 16, without bothering to verify whether the rule of proportionality applied to the triangles would give the results indicated thereon. Cf. pp. 42–47. For the consequences see note 36 below.

[10] *Ibid.*, p. 6.

[11] *Ibid.*, pp. 96–97.

are 1050, 2050, 3050 and 4050 for the 10, 20, 30 and 40 year groups respectively. The total money collected is the same in each case."[12]

We thus learn from Professor Kurtz himself that the only scientific approach consists of charging too little on more life units than there are in 100 machines, whereas the straight line method is unscientific enough to calculate the premium by reference to the right number! As his chart shows,[13] he has method 1 in mind, when concluding that his procedure is more scientific than the straight-line plan. The proper antithesis is method 3, which promptly demonstrates the pointless distortion. For the sake of comparison, Professor Kurtz's method may be expressed in the form:

$$(17) \qquad r_5(t) = 1 - \frac{t}{a + \frac{1}{2}}, \qquad 0 \leqq t \leqq \tfrac{1}{2};$$

$$= \frac{1}{a + \frac{1}{2}} \int_t^{n+1/2} M(\tau - \tfrac{1}{2}) d\tau, \qquad \tfrac{1}{2} \leqq t \leqq n + \tfrac{1}{2}.$$

This curve is plotted as a broken line in Figure 3 to show its divergence from the true method of replacement insurance for the case without interest.

6. *The Sinking-Fund and Annuity Methods*

Variations may be devised to correspond to any of the first four methods. For the counterpart of method 1 the premium is $P = i/(e^{ia} - 1)$, which leads to the formula:

$$(18) \qquad r_{6.1}(t) = \frac{e^{ia} - e^{it}}{e^{ia} - 1}, \qquad 0 \leqq t \leqq a.$$

This form also serves as a favorite antithesis. In practice the counterpart of method 2 is usually employed:

$$(19) \qquad r_{6.2}(t) = \frac{e^{ia} - e^{it}}{e^{ia} - 1} M(t), \qquad 0 \leqq t \leqq a.$$

The true replacement-insurance method for $i > 0$ would be:

$$(20) \qquad r_{6.3}(t) = M(t) - P \int_0^t M(\tau) e^{i(t-\tau)} d\tau + \int_0^t f(\tau) e^{i(t-\tau)} d\tau, \qquad 0 \leqq t \leqq n;$$

where P is taken from formula (16). The book value equals the difference between the cost of the machines still in service and the reserve,

[12] *Ibid.*, p. 112.

[13] *Ibid.*, p. 113, Figure 43.

to which premiums and interest are credited and benefits charged. The annuity formula is simpler and equivalent:

$$(21) \qquad r_{6.3}(t) = \phi \int_t^n M(\tau)e^{i(t-\tau)}d\tau; \qquad \frac{1}{\phi} = \int_0^n M(\tau)e^{-i\tau}d\tau.$$

The transformation of (20) into (21) is readily accomplished upon observing that $P=\phi-i$.[14] For equations (18) and (19) the equivalence of the sinking-fund and annuity methods is immediately apparent.

The discounting process can be applied to method 4 in the same way:

$$(22) \qquad r_{6.4}(t) = \int_0^{M(t)} \frac{e^{iM^{-1}(y)} - e^{it}}{e^{iM^{-1}(y)} - 1} dy = \int_t^n \frac{e^{i\tau} - e^{it}}{e^{i\tau} - 1} f(\tau)d\tau.$$

When i is the fair rate of return, this would be the equitable method of public-utility depreciation, if both the output and the operating expenses were constant.[15] No one ever uses it, of course. The device of dual rates is commonly employed to enhance the "rate" or unit price chargeable to consumers beyond a fair figure. According to the *trust-fund theory*, the sinking fund must be safely invested in behalf of the consumer, who is thus obliged not only to pay the "fair" or legal rate of return on the capital actually rendering a public service, but also to make good the deficiency in the earning power of the capital which he has already repaid.

It is unnecessary to present the formulae for the dual-rate methods, because the trust-fund theory is generally applied directly to the calculation of the "rate," without affecting the bookkeeping entries. The regulations usually endorse the "principle" that "the fund is a necessary part of the property of the utility and should not be excluded from the rate base; it may be considered as part of the working capital. A larger income must be permitted on the main operating property, in order to compensate for the deficiency of the earnings of this part of the invested capital."[16]

This conclusion is directly contradicted by the composite-book-value curve,[17] which shows that a very substantial part of the originally necessary capital is liquidated in the normal course of business and

[14] That the sinking-fund contribution is less than the depreciation, is overlooked by many writers. Despite the memorable failure of Dr. Richard Price—who, around 1770, insisted that John Bull could lift himself by his bootstraps through the magic of compound interest—the claim is still being advanced that the sinking-fund method is cheaper than any other.

[15] Cf. method 11 below.

[16] Perry Mason, *The Principles of Public Utility Depreciation*, American Accounting Association, Chicago, 1937, p. 81.

[17] Cf. *op. cit.* in note 1, Figure 1, p. 228.

ceases to render any public service thereafter. Even in the case of expansion and increasing replacement costs, large sums remain idle for a long time, until used for the purchase of necessary equipment.[18] In the meantime, these sums have no claim against the consumer.

Since the Bell Telephone System uses the straight-line method 3, it does not even credit the actual earnings of the sinking fund toward the "fair rate" calculated on the original(?) cost of the equipment in service. Federal Communications Commissioner Paul A. Walker's endorsement of the trust-fund theory[19] amounts merely to a demand that that be done. He does not question the validity of the theory itself. If capital can spill over from the original plant investment into a liquid fund, is that not conclusive evidence of its inability to render any public service?[20]

7. *The Diminishing-Balance Methods*

Four different approaches corresponding to methods 1 to 4 are again possible. A scrap value $s>0$ is essential. The book value of a single machine is for the first two cases:

$$(23) \qquad B(t) = s^{t/a} = e^{-kt}; \qquad k = -\frac{1}{a}\log_e s.$$

Omitting the first case, we obtain for the second:

$$(24) \qquad r_{7.2}(t) = \frac{1}{b}(s^{t/a} - s)M(t), \qquad b = 1 - s, \qquad 0 \leqq r \leqq a.$$

The third possibility is:

$$(25) \quad r_{7.3}(t) = \frac{1}{b}\left[e^{-kt} - s\int_0^t f(\tau)e^{k(\tau-t)}d\tau - sM(t)\right]; \qquad 0 \leqq t \leqq n.$$

If k, as determined from (23) is again applied, the essential condition $r_{7.3}(n) = 0$ will not be fulfilled. Such a method is therefore quite crude.

[18] *Ibid.*, Figure 3, p. 230.

[19] *New York Times*, April 2, 1938, p. 2, Summary of Telephone Report: "The accumulated reserves, as well as future additions thereto, should be administered by the company for the benefit of subscribers present and future."

[20] For a discussion in greater detail, cf. my paper, "The Principles of Public Utility Depreciation," *Accounting Review*, June, 1938, pp. 149–165.

As the present paper goes to press, I must acknowledge Prof. Ragnar Frisch's courtesy in lending me a copy of Prof. P. O. Pedersen's contribution "On the Depreciation of Public Utilities" (*Ingeniørvidenskabelige Skrifter*, B Nr. 12, pp. 69–99, Dansk Ingeniørforening, Copenhagen, 1934). Lack of time and space prevents me from doing justice to certain new and interesting aspects presented therein, e.g., effect of errors in life estimates, simultaneous scrapping of a whole plant, varying degrees of utilization, etc.

The defect can be remedied, however, by using a rate k given by that condition. The result is the insurance premium (for $i=0$), which must be charged on the basis of *composite* book values, instead of on cost as in method 3. The form of (25) remains unchanged; its new meaning may be denoted by $r_{7.30}(t)$.

For the fourth variety of the diminishing-balance method, $B(t)=s^{t/M^{-1}(y)}$, which leads to the formula:

$$r_{7.4}(t) = \frac{1}{b}\int_0^{M(t)} (s^{t/M^{-1}(y)} - s)dy, \qquad 0 \leqq t \leqq n. \tag{26}$$

Of these methods, the only one occasionally employed in practice is 7.2. American accounting writers usually mention that it is used more extensively in Great Britain. The comment is also generally made that the method counteracts the increase in maintenance charges, as a machine grows older.

8. *The Sum-of-the-Year-Digits Method*

The depreciation charge is apportioned in accordance with a declining arithmetic series. For instance, if $a=8$, the first year's charge would be 8/36, the next year's 7/36 and that of the last year 1/36. The procedure amounts to calculating the remaining area, at the time t, of the shaded triangles of Figure 1. Thus for the counterpart of method 2, there is obtained:

$$r_{8.2}(t) = \left(1 - \frac{t}{a}\right)^2 M(t), \qquad 0 \leqq t \leqq a. \tag{27}$$

That is no doubt the only form in which the method might possibly occur. I have never seen it applied, but many accounting textbooks mention it. Presumably, it also means to counteract the declining efficiency of a machine.

9. *The Retirement Method*

Depreciation is disregarded altogether, so that no reserve for depreciation appears on the books. The balance of the plant account is also the book value of the machines in service. The cost of scrapped machines is deleted promptly by direct charge to operations:

$$r_9(t) = M(t), \qquad 0 \leqq t \leqq n. \tag{28}$$

So-called retirement reserves are created merely to smooth out the fluctuations between theoretical and actual rates of replacement. Such reserves seldom exceed one per cent, of the plant account, and are the best evidence that a large replacement fund is unnecessary.

The retirement method is used extensively in the public-utility field, where it combines the advantages of an undepreciated rate base with freedom from both legal and financial restrictions on dividend payments. Superfluous capital can be paid out without hindrance, because it is legally classified as surplus or undivided profit, and because the rate base will not be reduced by the distribution. When another method is used, the same sums are classified as a necessary replacement fund, the distribution of which is contrary to law, because such action would impair the corporate capital! This legal inconsistency is no doubt responsible for the trust-fund theory. The correct principle appears to be the one which I have elsewhere called "the principle of good management,"[21] according to which the company shall neither hoard idle funds, nor part with or fail to raise any, which can be productively reinvested. For public utilities, this principle must of course be applied from the consumer's viewpoint of what constitutes idleness.

10. *The Canning Method*

The method of Professor John B. Canning[22] consists of the simpler and therefore more practical features of the Taylor-Hotelling method,[23] which was already examined.[24] Whereas methods 1 to 9 are concerned only with the distribution of original cost over the life of a machine, Professor Canning also considers "outlays . . . that will most probably have to be incurred as a direct . . . consequence of . . . ownership and operation, if the machine's service is to be had most economically."[25] There is reason to believe that this means chiefly repairs and maintenance, i.e., current physical upkeep. Professor Hotelling's operating expenses are probably all-inclusive, although he does not define them. Both methods look alike when the rate of interest and the scrap value are considered constant:

$$B(t) = \int_t^T [wQ(\tau) - E(\tau)]e^{i(t-\tau)}d\tau + se^{i(t-T)}. \tag{29}$$

[21] See *The Nature of Dividends*, New York, 1935, p. 9, pp. 175 *et seq.* Not to distribute reinvestable funds may mean in practice either no distribution at all, or stock dividends, or an offsetting combination of cash dividends and subscription rights.

[22] *The Economics of Accountancy*, Ronald Press Co., New York, 1929.

[23] J. S. Taylor, "A Statistical Theory of Depreciation," *Journal of the American Statistical Association*, December, 1923, pp. 1010–1023; Harold Hotelling, "A General Mathematical Theory of Depreciation," *ibid.*, September, 1925, pp. 340–353.

[24] See *op. cit.* in note 1, pp. 234–239.

[25] Canning, p. 291.

In this notation, $B(t)$ = book value of a single machine, w = unit cost including time cost, $Q(t)$ = rate of production, $E(t)$ = expenses, s = scrap value, and T = date of scrapping. Professor Canning considers only $B(t)$ and w as unknown so that a second equation expressing the original cost $B(0)$ is sufficient. In the Hotelling theory T is to be determined simultaneously from the additional relation $dB(t)/dT=0$.

Professor Canning holds that "no formula . . . will show exactly when an asset is worn out, but the good ones [i.e., his own with or without interest] will . . . show that a wrong estimate of T has been made before any substantial real loss has been incurred."[26] They "serve notice . . . by finding negative valuations long before the end of the Tth period."[27] These remarks refer to his example G,[28] where a sharply declining rate of production coupled with sharply increasing outlays soon leads to negative book values.

That the emergence of negative values need not give timely warning, may be inferred from example D,[29] where the rate of production is constant. An outlay of \$25 is there made during the tenth and last year of life on a \$100 machine having a book value of \$7.47 at $t=9$. Unless the profit (after all other expenses including depreciation, but before the outlay) is more than 334.7 per cent, of the investment, it would also have been better to scrap the machine. As in Professor Hotelling's theory, the real market price of the product is omitted from the problem.

Methods similar to Professor Canning's are sometimes employed, especially in the simplified form $i=0$, which he really prefers. For many machines the formula becomes in practice a variation of method 2:

$$r_{10.2}(t) = \frac{M(t)}{b}\int_t^a [wQ(\tau) - E_C(\tau)]d\tau, \qquad 0 \leqq t \leqq a. \tag{30}$$

Under this procedure the cost of repairs, etc., is charged to the reserve and not to operations. The periodic debit to operations and credit to the reserve is $wQ(t)\Delta y$ per machine, but journal entries analogous to (8) or (9) are also made when scrapping takes place before or after the average age a. A difference arises in each case between the actual repair charges and the amounts previously credited to the reserve to provide for repairs. Unless such balances are promptly traced and closed out to operations, an error in average repair estimates will have a cumulative effect upon the book value of a composite plant. When

[26] *Ibid.*, p. 264.
[27] *Ibid.*
[28] *Ibid.*, p. 349.
[29] *Ibid*, p. 346.

this danger is overlooked, as it sometimes is, the official description of the depreciation method may become misleading.

11. *The Theoretical Public-Utility Method*

Although no depreciation theory may be described as the general[30] one, that of Professor Hotelling appears to be the correct solution of the specific problem presented by regulated monopolies. The only change necessary is to substitute the legal rate of return p for the rate of interest i, which he would employ in every instance. If his three equations are so revised, the simultaneous solution yields the correct date of scrapping and the correct unit cost plus profit, i.e., the fair "rate" chargeable for the services of a single machine. In terms of many machines installed at the same time, the method will then amount to:

$$(31)\qquad c(t) = \int_0^{M(t)} B_p(y, t)dy, \qquad 0 \leqq t \leqq n,$$

where B_p means formula (29) with p substituted for i. The inclusion of y indicates that the functions T, w, Q, and E differ from machine to machine. The last two vary of course also with time. The equation $dB(y, t)/dT = 0$ now furnishes the mortality curve $T = M^{-1}(y)$, which must be employed if the service of all machines "is to be had most economically."[31] We therefore obtain an elaboration of method 4:

$$(32)\quad r_{11.4}(t) = \frac{1}{\beta}\int_0^{M(t)} \int_t^{M^{-1}(y)} [w(y)Q(y, \tau) - E_H(y, \tau)]e^{p(t-\tau)}d\tau dy,$$

where β is the factor of $r(t)$ in equation (3), when p is read for i. If Q and E_H are independent of time, $w(y)Q(y) - E_H(y) = \beta p/(1 - e^{-pM^{-1}(y)})$ and formula (32) reduces to (22). In that case mortality must be a physical rather than a value phenomenon.

That the theory of public-utility depreciation can be definitely stated in mathematical terms does not mean that it is readily applicable in practice. Some of the legal and financial obstacles have already been mentioned.[32] Practical difficulties are the lack of foresight, the clerical expense of keeping the voluminous records required to make deferred adjustments by hindsight and the complexity of the calculations which arise. To give an indication of the latter, I shall cite the simplest example I was able to construct. Bearing in mind the three simultaneous equations, let:

[30] Cf. note 23.

[31] Cf. note 25.

[32] Cf. also my *op. cit.* in note 20 and my earlier paper, "The Law of Goodwill," *Accounting Review*, December, 1936, pp. 317–329.

$$(33) \qquad w(y)Q(y, t) = [ps + E_H(y, T)]e^{q(T-t)}$$

and

$$(34) \qquad E_H(y, t) = \frac{q + p - s(qe^{-pT} + pe^{qT})}{e^{(k+q)T} - \dfrac{k+q}{k-p}e^{(k-p)T} + \dfrac{q+p}{k-p}} e^{kt}.$$

Now, if $k=3p$, $q=p$, and $T=2a(1-y)$, the solution of (32) is:

$$(35) \qquad r_{11.4}(t) = \frac{1}{4ap\alpha\beta}\left[\frac{s(\alpha^2 + \alpha^4)}{z} + 2\alpha^4 \log z - (\alpha^2 - 1)^2 \frac{1 - sz}{z^2 - 1} - (\alpha^4 - 1) \log \frac{(z+1)^{s+1}}{(z-1)^{s-1}}\right]_\alpha^\epsilon.$$

In this notation, z is the variable of integration $e^{2ap(1-y)}$, for which the limits $\epsilon = e^{2ap}$ and $\alpha = e^{pt}$ have not yet been substituted. For the special case $a=8$, $p=.07$ and $s=.1$, the graph is shown in Figure 4, although it has no connection with the mortality curve (5), but refers to $M(t)=1-t/16$. For the curve (5), $M^{-1}(y)$ is the second root of a quartic equation, which is already too involved for an example. An actual case is far worse.[33] A less simple relationship of k, q, and p also increases the difficulties.

III

One general measure of the differences in depreciation methods is the ultimate composite-book-value level determined in the absence of scrap value, expansion, or change in replacement costs. The general formula of the level is:[34]

$$(36) \qquad R(\infty) = \frac{1}{a}\int_0^n r(t)dt.$$

It is of interest to see how differences in the shape of mortality curves affect the ultimate levels. All observed mortality curves[35] normalized to the same area a lie within the extremes:

$$(37) \qquad M(t) = 1 - t/2a, \qquad M^{-1}(y) = 2a(1 - y),$$

and

$$(38) \qquad M(t) = 1, \qquad M^{-1}(y) = a.$$

[33] See the frequency distributions fitted to actual data by Professor Kurtz, *op. cit.*, in note 3, pp. 103–106.

[34] By *ibid.* in note 1, formula (17), when $x=0$, $q=0$, $s=0$ and $t=\infty$.

[35] Cf. seven mortality curves calculated by Professor Kurtz, *ibid.*, in note 3, p. 74.

Upon inserting (6) into (36) we obtain:

$$(39) \qquad R_1(\infty) = \frac{1}{a}\int_0^a (1 - t/a)dt = \frac{1}{2}\,.$$

The method itself requires the use of (38), so that (37) cannot be applied. The levels of the next three methods may be written in their general form, placed between the extremes (37) and (38) in that order:

$$(40) \qquad \frac{5}{12} < R_2(\infty) = \frac{1}{a}\int_0^a (1 - t/a)M(t)dt \qquad < \frac{1}{2},$$

$$(41) \qquad \frac{2}{3} > R_3(\infty) = \frac{1}{a^2}\int_0^n \int_t^n M(\tau)d\tau dt \qquad > \frac{1}{2},$$

$$(42)^{36} \qquad \frac{1}{2} = R_4(\infty) = \frac{1}{a}\int_0^n \int_t^n (1 - t/\tau)f(\tau)d\tau dt = \frac{1}{2}\,.$$

[36] The mistake of considering methods 3 and 4 as equivalent frustrates Professor Kurtz's efforts to obtain the result (42) by valid mathematical processes, because the proper use of his Table VIII (cf. note 9) leads to (41). He must get a stable level of ½ however. A shaded triangle having half the area of a layer in Figure 4 above, it is evident enough that the ultimate static composite level is 50 per cent for that method, no matter what the shape of the mortality curve may be. Professor Kurtz cuts this Gordian knot by saying that "each group of survivors is multiplied by the per cent remainder service life" (p. 78) taken from Table VIII, which is equivalent to formula (13) above, except for the difference between discrete and continuous functions. In his Table XX he thus gets arithmetically the results:

$$\text{(K)} \qquad r_K(t) = \frac{1}{a} M(t) \int_t^n M(\tau)d\tau;$$

$$\text{(42K)} \qquad \frac{1}{2} = R_K(\infty) = \frac{1}{a^2}\int_0^n M(t) \int_t^n M(\tau)d\tau dt = \frac{1}{2}.$$

This is considered as "convincing evidence of the soundness of the method . . . the logic . . . the accuracy . . . their correct joint use . . . a verification of the entire preceding analysis and a demonstration of the inherent unity of the body of principles presented" (pp. 5–6). Actually, he proves nothing but the elementary rule $\int v dv = v^2/2$, which is not applicable, because there is no warrant in theory to multiply only the survivors of sucessive renewal groups by the corresponding percentages of Table VIII. To do so would have been proper only if that table had been calculated per centum of the number of life units originally contained only in the machines surviving at any time t. In fact the percentages are based upon all life units contained in all machines at $t=0$.

It follows that before the renewal rate $u(t)$ is stabilized at $1/a$, the true damped waves of composite remainder cost for method 4 differ from those to which Professor Kurtz's empirical equations have been fitted in his Figures 24–30. That is to say, $r_4(t) \neq r_K(t)$ as shown in my Figure 4 and therefore $R_4(t) \neq R_K(t)$, even

The formulae for the remaining methods will be omitted. Results for simple concrete assumptions are given in Table 1.

TABLE 1

<table>
<tr><th rowspan="2">Method</th><th colspan="2">Level for Mortality Curve</th><th rowspan="2" colspan="2">Other Assumptions</th></tr>
<tr><th>(37)</th><th>(38)</th></tr>
<tr><td>5</td><td>0.688</td><td>0.531</td><td colspan="2">$a=8$</td></tr>
<tr><td>6.1
6.2
6.3
6.4</td><td>—
0.451
0.724
0.564</td><td>0.546
0.546
0.546
0.546</td><td colspan="2">$a=8,\ i=0.07$</td></tr>
<tr><td>7.2
7.3
7.30
7.4</td><td>0.281
0.329
0.442
0.323</td><td>0.323
0.323
0.323
0.323</td><td colspan="2">$s=0.1$</td></tr>
<tr><td>8.2
9</td><td>0.292
1.000</td><td>0.333
1.000</td><td colspan="2">none</td></tr>
<tr><td>10.2</td><td>0.301</td><td>0.343</td><td rowspan="3">$a=8$
$s=.1$</td><td>See formulae (33) and (34) for $T=8$.
$wQ(t)=w(\frac{1}{2})Q(\frac{1}{2},\ 0)e^{-.07t}$
$E_C(t)=E_H(\frac{1}{2},\ 0)(e^{.21t}-1)$</td></tr>
<tr><td>10.3</td><td>0.702</td><td>0.417</td><td>Anticipated from part IV.
$E_C(y,\ t)=E_H(y,\ 0)(e^{.21t}-1)$, or
$E_C(t)=E_H(\frac{1}{2},\ 0)(e^{.21t}-1)$,
corresponding to $T=16(1-y)$ or $T=8$.</td></tr>
<tr><td>11.4</td><td>0.387</td><td>0.377</td><td>Formulae (33) and (34) with
$T=16(1-y)$ or $T=8$.</td></tr>
</table>

The levels corresponding to (38) do not really exist, but are mere averages, because no damping can take place. All composite-book-value curves remain forever similar to the edge of a saw, dropping gradually from unity to zero and jumping back suddenly every a years. Damping is intensified as the shape of a mortality curve approaches the opposite extreme (37).

though $R_4(\infty)=R_K(\infty)$. In addition, simple damped sine functions do not fit the oscillations of either the right or the wrong data.

Why the stable level (42K) was calculated at all is not apparent. Professor Kurtz abandons it immediately after declaring that it is "basic in this study" (p. 90). His "normal" insurance reserves for $i=0$ are not $\frac{1}{2}$, but differ only slightly from $1-R_3(\infty)$, because (17) is fairly close to (13), as seen in Figure 3 above. The enormous volume of arithmetical calculation performed to determine these levels by "causing the property to pass through its early history or by a 'reverse liquidation process' " (p. 125) is also wholly unnecessary. See formula (36) above.

The table is valid only in the absence of expansion and only for static replacement costs. Steady increases in the purchase price of new machines and in the number of machines in service lead to higher levels per centum of the rising original cost. Irregular changes perpetuate the fluctuations.

IV

Another significant test of differences in depreciation methods is the degree of fluctuation which they introduce into the net profit. For brevity, I shall omit an adequate presentation and merely call attention to the samples of composite depreciation curves shown in Figures 2 and 4 of my survey of the basic theory.[37] The methods there numbered from 1 to 4 are identified by the subscripts 9, 3, 6.3, and 7.3 respectively in the present paper.

A study of the question, what method is most suitable for competitive enterprise, must start from the principle already established that the depreciation problem is fundamentally indeterminate. The theory of economic life can furnish only the average life-span. Within that period, the wearing value may be written down in innumerable different ways, some of which look more reasonable than others. In such circumstances, the choice remains in part a matter of opinion; some progress can be made, however, toward accomplishing practical objectives. The principal one is that of facilitating forecasting, i.e., appraisal by the stock market. Related to it is the task of developing a method of cost accounting, which will be most informative to the management. For both purposes, a method is required, which distributes all predictable costs as evenly or proportionately as possible.

From this strictly practical viewpoint, it is necessary, first of all, to give up the idea that the figure at which capital assets are stated in the balance sheet means anything in particular, except *at best* "actual expenditure not hitherto charged against profits."[38] Similarly, *lasciate*

[37] *Op. cit.* in note 1, pp. 229, 231.

[38] That is the best theoretical, i.e., wishful, definition available (Anonymous: "Goodwill and Advertising," *Accountant*, London, February 18, 1914, pp. 287 *et seq.*). Mr. May concludes merely that a balance sheet "is a highly technical production, the significance of which is severely limited and has in the past often been greatly overrated" (*op. cit.* in note 2, p. 20).

A Statement of Accounting Principles, published under the auspices of the American Institute of Accountants (New York, 1938) still contains the hoary platitude that "a balance sheet is a statement which purports to exhibit the financial condition of the business" (p. 55). To understand how little this means, it is necessary to note that "practice has in the course of time hardened into a set of general conventions" (p. 56) expounded in 110 pages. The gist of these is that, subject to legal limitations and after careful consideration, an accountant should do what he thinks best, at least "unless and until sound

ogni speranza[39] that the theory of economic life[40] can soon be applied in practice. When to scrap a machine will have to remain for the present a matter of technical judgment, aided perhaps by records of output and expenditure kept for isolated samples. If these major aims of theory are abandoned, it immediately follows that there is no point in calculating interest. By eliminating that refinement also, a great practical advantage is gained; it becomes possible to measure exhaustion of service-capacity in terms of production units instead of time.[41]

Once this stage is reached, the choice falls automatically on the true straight-line method 3, because it will charge a constant insurance premium per unit. That charge will not be affected by either the age of the plant or a variable rate of expansion $x(t)$. When replacement costs increase at the rate $q(t)$, the premium increases only in proportion to the average change in the total cost of the plant arising from that source:

$$(43) \qquad P(x, q, f, t) = \frac{1}{a} R_9(x, q, t)e^{-\int_0^t x(\tau)d\tau}; \quad P(x, 0, f, t) = \frac{1}{a} \cdot$$

If desired, most of the fluctuation due to replacement costs can be

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business judgment dictates" (p. 39 and to the same effect elsewhere) that he do something else for reasons not divulged. For details of "such abnegation, such subservience," see W. A. Paton, "Comments on 'A Statement of Accounting Principles'," *Journal of Accountancy*, March, 1938, pp. 196 *et seq.* Also Howard C. Greer, "What Are Accepted Principles of Accounting?" *Accounting Review*, March, 1938, pp. 24–31.

[39] Inscription above the portal to Dante's *Inferno*.

[40] Several months' intensive study of this problem convinces me that it is far more complicated than Professors Taylor and Hotelling conceived it to be fifteen years ago, or myself for that matter, as late as last year. The disconnected comments made in my "Theory" are in order, as far as they go, but they fail to tell the whole story.

Depreciation is merely a distorted shadow cast upon the books by replacement policies. The theory of the latter appears to be essentially a theory of scarcity. As many different rules of replacement can be had as there are ingredients of production or possible combinations of such scarce ingredients. Since the relative weights of different ingredients vary even within a single field of endeavor, each individual enterprise will have its own rule. The sole exception is the case where the market price is fixed by law or otherwise. This rigid limitation overrules the influence of all elastic scarcities.

At the moment when the galleys of the present article are before me (May 25), my MS on economic life is in the hands of Professor Hotelling, to whom I submitted it in recognition of his priority in this field. It is hoped to publish this in the near future.

[41] It is the omission of interest which makes it possible to avoid guessing the future course of production. Otherwise the difficulty is simply shifted to the discount factor. Accounting writers realize this well enough, but a few others, including Professor Kurtz, give no indication of doing so.

eliminated by adopting "Stabilized Accounting,"[42] i.e., keeping books not in dollars, but in terms of an index of general purchasing power. Some variation will remain, because the individual indexes of many different types of equipment always show dispersion.

The distribution of costs of acquisition can thus be made fairly even. Next, let us extend the insurance principle to expenses which can be predicted with a reasonable degree of accuracy. Repairs and maintenance are a good illustration. For a large number of machines installed at the same time, such expenditures form the frequency distribution:

$$\eta(t) = \int_0^{M(t)} E_C(y, t)dy, \tag{44}$$

which may be treated in the same way as that of replacement. The premium covering both risks will be:

$$P(x, 0, f + \eta, t) = \frac{1}{a}\left[1 + \int_0^n \eta(\tau)d\tau\right]. \tag{45}$$

The search for the method most suitable for practical purposes thus leads to a second application of Professor Canning's single-machine theory:

$$r_{10.3}(t) = \int_t^n \left[PM(\tau) - \eta(\tau)\right] d\tau. \tag{46}$$

To what extent expenses are to be included, depends upon the accuracy attainable in forecasting them. The aim is to even out the fluctuations, but it is also essential that the differences between the premiums credited to the reserve and the actual expenses charged thereto be adjusted by hindsight from time to time for suitable groups. If foresight is too poor, the adjusting entries may cause greater fluctuation in the time shape of expenses, than if the least predictable items had been excluded from the averaging process and charged currently instead. For a large and *mature* enterprise a substantial degree of equalization will occur even without any special effort. For this reason the Bell System is apparently content to use only the simple method 3. Its special problems call for apportionment on the basis of time, despite cyclical changes in the volume of production. Where wear and tear from use is the principal cause of capital consumption, a distribution of cost over production units seems preferable. This changes only the meaning of t and the form of all functions dependent on it, but not the symbolic formulae.

[42] Title of a book by Henry W. Sweeney, Harper & Bros., New York, 1936.

As already stated, no variety of method 3 should be applied in practice, unless a company maintains a competent statistical department. Others had best adhere to the old stand-bys, methods 2 and 7.2, or perhaps 10.2. Most American companies do indeed use method 2, generally in terms of time, but occasionally on a production basis. Some concession to method 3 is frequently made by disregarding the "loss on capital assets" (8) for purposes of cost accounting. This item then appears in the financial, instead of the operating section of the profit-and-loss account.

It is perhaps ironical that the one enterprise generally known to use method 3 on the proper actuarial basis is a public utility. Still, an unbiased appraisal of the enormous difficulties connected with the theoretically true special method 11.4 may well lead to the conclusion that the straight-line method is most suitable for that field also. If it be found easier to adjust the nominal rate of fair return than the depreciation method, that is the obvious alternative. The essential consideration remains the actual rate of return.

When the expansion rate $x(t)$ is fairly steady, the simple straight-line method 3 can be made more flexible by varying the premium P, which would ordinarily be $1/a$, within the limits:

$$\chi - x \leqq P \leqq \chi, \tag{47}$$

where χ is the asymptote $u(x, \infty)e^{-x\infty}$ of the renewal rate. Any such premium will be linearly but inversely related to the ultimate composite-wearing-value level per centum:[43]

$$R(\infty) = \frac{\chi - P}{x}. \tag{48}$$

The straight-line method could thus be used by a mature and steadily expanding company in such a way as to duplicate closely the true book-value level determined by a process of testing or sampling based on the theory of method 11.4. Regardless of this suggestion, a periodic test of the book-value level should be recognized as an essential element of regulatory policy. The profit-and-loss account of a public utility may be written in the simplified and purely symbolic form:

$$Z = A + P + iB + p(C + W - B), \tag{49}$$

wherein Z = gross revenue claimed to be fair, A = expenses not provided for in the premium, B = borrowed capital (including preferred

[43] By *op. cit.* in note 1, formula (34). Replace $b = 1 - s$ by unity and delete sx. See also formula (32) and Figures 1 and 3, *ibid.*

stock), iB = its hire, p = fair rate of return, and W = rate-base assets other than the plant, e.g., working capital. By substituting the results of the test for the actual premium P and the actual book value C, it can be readily ascertained to what extent Z is too large or too small. The development and application of such a test should have been among the principal aims of the recent investigation of telephone companies by the Federal Communications Commission.

On the whole, then, accountants are not greatly to blame for their depreciation methods, as far as competitive business is concerned. In the public-utility field, on the other hand, they are distinctly reluctant to face the facts and inclined to continue passing them off as matters of opinion. To call attention to what remains undone in practice, it seems appropriate to close this paper by quoting a few "accounting principles" on public-utility depreciation:

> The question of the adequacy of so-called retirement and similar provisions for depreciation can be answered only from an examination of the total amounts actually provided for depreciation and maintenance over a considerable period. It is the sum of the two which is to be regarded as adequate or inadequate. . . . In the opinion of many competent observers, retirement methods do in fact result in inadequate charges for depreciation, especially when considered with respect to the maintenance of the original investment.[44]
>
> The provision for depreciation . . . is probably the most controversial subject related to the accounts of public utilities . . . on which professional accountants are expected to have definite opinions. . . . Within the committee there are definite differences of thought. . . . Those who favor the straight-line method . . . point critical fingers at a utility company which shows in its income-tax return a deduction for depreciation computed by the straight-line method and . . . in its published financial statements . . . a smaller amount as the provision for retirements. . . . They are distrustful because the . . . officials of the utility use their independent judgment . . . without being bound by any fixed formula of estimated service life . . . The retirement advocate does not see any need for . . . reserves such as would measure . . . lack of newness . . . He is not impressed by the argument that the officials of a utility must be compelled to provide for depreciation in amounts other than those dictated by their own judgment, merely because some other person has estimated a list of service lives for *pieces* of the very system which the officials view as a unit . . . He reasons that taxable income is a statutory concept, not a determination based on economic principles. . . . One may dissent very strongly about the wisdom of a statute and . . . be completely obedient to it. . . . The presentation of the arguments on both sides of the controversy does not provide a solution to the problem. . . . The committee has taken no action within the current year, on this subject.[45]
>
> It has been said that we owe our great railroad facilities . . . in a large measure to unsound finance; but if it be held that depreciation provisions are an essential element of sound railroad accounting, then unsound accounting must share in the responsibility for the tremendous economic development that has taken

[44] *A Statement of Accounting Principles*, pp. 31–32. Cited in note 38.

[45] "Report of the Special Committee on Public-Utility Accounting," *Midyear Review*, American Institute of Accountants, 1938, pp. 66–69.

place since railroad enterprises were first begun. . . . It is no doubt true that, as a result of the accounting methods followed, large amounts of capital have been lost by investors . . . This, however, merely emphasizes the truth too often ignored . . . that, in the aggregate, the community pays only a relatively small return to capital for the amount invested, and that it is the community that is the one sure gainer therefrom. . . . The community can well afford to allow the few who meet with unusual success to receive and retain substantial rewards as a part of the price which it pays for all the capital invested.[46]

These views do indeed sound as if accounting were only what Mr. May says it is.[47]

New York, N. Y.

APPENDIX A

Professor John B. Canning, who refereed this paper, was good enough to propound a number of questions, which he thought many readers might wish to ask. Professor Frisch decided to have these questions and my answers appended to the paper. They follow:

1. Anent the "true" straight-line method:—Is this "true" in any sense except that: (1) it is free of certain objections to which 1 and 2 are open; and (2) it conforms to certain (not exclusive) provisions of *Income Tax Regulations?* On page 243 the author himself notes an objectionable feature of the "true" method. To refer to the method in question as a "third" method (or by some other noncommittal term) and then set forth its formulae and their properties would in no way weaken the high substantive merit of the section and would be more likely to keep his critics on the rails (see his first paragraph).——*Answer:* The description "true straight-line method" was not intended to imply that this is the "true" method, but merely that it is the only "truly straight" one. In other words, method 3 makes a charge corresponding to the asymptote of the renewal rate, which is a horizontal line in the static case. The charge per machine per unit of time (or per unit of output, if preferred) is constant even when the plant grows at a variable rate. It is similarly independent of the age of the machines. See Section IV.

2. The elementary single-machine book-value formula of which his composite functions are constructed is of the form $v=a_0+a_1t$. One may suggest another, namely, $V=A_0+A_1t+A_2t^2$. If the constants in both are to be fitted, by whatever criteria, then, in no case, can his formulae give results superior to those of the alternative, for his formula becomes merely the limiting case (in fitting) in which A_2 approaches the limit 0.——*Answer:* The only curve to be fitted is the frequency distribution $f(t)$, the nature of which I left unspecified, in order that all

[46] May, *op. cit.* in note 2, part III, March, 1936, p. 175.

[47] Cf. statement identified in note 2.

$r(t)$ formulae be entirely general. The example (5) serves only the purpose of graphic illustration. No occasion arises for fitting the curves suggested and certainly they are not the elements of *my* composite functions. In so far as any similar expression occurs in the text, it simply defines a depreciation method, which I did in no sense originate. I am merely the translator of debits and credits. Accordingly *my* formulae are not supposed to be superior, but merely correct in reflecting a certain method, which may be open to many objections.

3. Can $r(t)=$ "undepreciated remainder per *centum* . . ."? By (6), we have $r_1(t)=1-t/a$, in which $0 \leqq t \leqq a$. Hence $r_1(t)$ has unity as a maximum at $t=0$. Isn't $r_1(t)$ rather an "undepreciated remainder per *unit* . . ."?——*Answer:* Stand corrected. It should be clearly understood, however, that the *unit* is the whole plant and not a single machine. To cite the classic illustration, 248,707 new telephone poles would be represented by unity. Single-machine formulae would be for instance the integrands in the second form of equation (13) or the first form of (14).

4. In connection with the accountant's method it might be well—say by footnote—to distinguish between formal charges (adjusting entries) to (manufacturing) operations and definitive charges against the income of the year in which the operations occur. In algebraic effect, an increment of the formal depreciation debit is reversed by a credit to income (*via* the goods-in-process and finished-goods inventories). This is not well understood by statisticians. It derives its importance from changes in inventory carryovers—especially in the case of concerns that, in effect, speculate in their own products (see ECONOMETRICA, Vol. 1, pp. 54–55, notes 3 and 5).——*Answer:* That is true enough, but has no bearing upon the book value or wearing value of the plant, with which I am alone concerned. Manipulation of inventories is an interesting subject, but it has no connection with depreciation, even if depreciation figures are used for the purpose in preference to others.

5. Method 6.4. Wouldn't the remark that "this would be the equitable method of public-utility depreciation, if both the output and the operating expenses were constant," be more apropos of the "theoretical public-utility method" here referred to for comparison?——*Answer:* I don't think so. Formula (22) is a special case of (32) and the latter in turn is the special case where output and expenses vary with the age of the machine, but economic conditions are otherwise static. General dynamic conditions upset the theory of economic life inherent in the Hotelling approach. The life spans of successive replacements can then be calculated only backward, step by step, i.e., in the reverse order of chronology, beginning with the last machine, which will never be re-

placed (and probably has not even been invented as yet). This being absurd, economic life must be guessed within a reasonable margin, as Professor Canning holds in substance. The public-utility problem would still be definite, however, if errors of foresight (apart from those inherent in any decision to scrap) were corrected by hindsight, i.e., by charging or crediting a "consumers' surplus." The analysis of single and composite chains of replacement is among the topics of my latest study. See note 40.

6. Methods 7 and 8. Do the comments that these methods counteract the increase in maintenance charges or decrease in efficiency have enough merit to warrant inclusion? The same could be said of an unlimited number of other formulae, e.g., of other members of the same family of curves. Any particular declining series of charges can be appropriate (given the criteria) only by accident. Cf. Hatfield, *Accounting*, pp. 153–154.——*Answer:* These are merely historical comments not intended to imply that either method can possibly be correct, except in case of a highly improbable accident, capable of verification only in the public-utility field.

7. Sentence after quotation identified by note 25. Properly inferred for many items, for the reasons suggested on pp. 273 and 298 of the book referred to. But for the inclusion of others (e.g., annual license fees) see p. 201.——*Answer:* Stand corrected.

8. Page 251, 3rd paragraph. Point well made. Moreover, an asset may have a positive value at or immediately before salvage sale and have, earlier, a quite proper negative book value. This would be the case, for example, if demolition and removal charges are expected numerically to exceed the receipt from salvage sales.——*Answer:* True enough from a practical viewpoint. In theory, the continuous time-shape of depreciation is indeterminate, as shown by the fact that depreciation does not enter into the theory of capital value at all. It is therefore difficult to say what would or would not be "quite proper," except by reference to such practical criteria as suggested in Part IV.

9. Isn't the "real market price" of the product a "will o' the wisp" save in the instance of a few commodities and services at a few of their stages between extraction and consumption? When numerous types of assets are jointly employed in the concurrent production of a varied "line" of products and when each concern's "line" changes in ways impossible of prediction n years hence, is it possible in any useful realistic sense to speak of *the price* of the *product?* (See *Economics of Accountancy*, p. 232 and p. 299, note, and the references there cited.) ——*Answer:* The theory of economic life could never be developed without recognizing that the omission of the market price from the problem is incorrect. In practice I agree whole-heartedly, as seen in

Section IV. But it is better to keep theory a step ahead of common sense than to let it trail behind, as I am afraid it does, in some respects, in the accounting field.

10. To be sure, we have not yet developed a theory of economic life that omits the market price of the product from the problem—but that is hardly conclusive evidence that it cannot be done. If economic life, in fact, proceeds despite such an omission, may not a theory consonant with the omission be devised? In any event, when theory deviates from practice, one needs to know whether the direction is "ahead" or just "off" before one can be sure that theory is leading practice or merely deserting it.——*Answer:* That, of course, is *the* question! It will be easier to discuss it, after my tentative theory of economic life is published (see note 40). The late President Wilson's advice that any theorist should have a sign on his desk reading "Don't be a damned fool!" is valuable indeed, but not everyone can hope to match Professor Canning's skill in devising a "realistic theory" in one fell swoop. I find it easier to begin by turning the sign to the wall and investigating, first of all, what we ought to do, if we could. Then, turning the sign front again, the excess of theory can be readily whittled down by an application of realism. In the present instance, the results happen to be almost identical. The seemingly useless trip into the realm of theory nevertheless gives an added feeling of security, which I would be loath to miss.

APPENDIX B

Symbols employed frequently, or far from the spot where they were introduced, are redefined in the following glossary:

t = a variable representing time, or a more suitable measure of the exhaustion of service capacity. Alternatives practicable only when interest is omitted.

τ = a variable of integration corresponding to t.

y = a variable of integration denoting divergence in individual behavior-characteristics of machines of similar type.

n = maximal life span, i.e., limit of longevity, measured in the same unit as t.

a = average life-expectancy of any new machine in same unit as t.

s = scrap value, divided by original cost.

b = original cost, less scrap value, divided by original cost.

β = original cost, less present worth of all scrap values of a large number of machines installed together, divided by original cost.

$f(t)$ = frequency distribution (of disappearance or scrapping) of a large number of machines installed together.

$M(t)$ = mortality or survival curve, i.e., cumulative frequency distribution.

$r(t)$ = undepreciated remainder of wearing value of a large number of machines installed together, divided by their original wearing value.

$r'(t)$ = rate of change, i.e., rate of depreciation of $r(t)$.

$c(t)$ = undepreciated cost of many machines installed together, divided by their original cost.

$M^{-1}(y)$ = inverted mortality formula, expressing life of the yth machine out of a very large total number, when their lives are extended horizontally and arrayed from top to bottom, from the shortest to the longest.

$T = a$ = economic life of any machine, when all behave alike.

$T = M^{-1}(y)$ = ditto, when individual characteristics differ.

$B(t)$ = book value of any single machine, when all behave alike.

$B(y, t)$ = ditto, when individual characteristics differ.

$\left.\begin{matrix} w \\ w(y) \end{matrix}\right\}$ = unit cost of product, including time cost. Same distinction.

$\left.\begin{matrix} Q(\tau) \\ Q(y, \tau) \end{matrix}\right\}$ = rate of production of any single machine at age τ.

$\left.\begin{matrix} E(\tau) \\ E(y, \tau) \end{matrix}\right\}$ = rate of operating expenses of any single machine at age τ. Indexes C and H identify Canning and Hotelling concepts respectively.

P = insurance premium per machine per unit of time or some alternative unit.

$R(\infty)$ = ultimate level of composite wearing value, when a plant has been maintained at a constant number of machines for a theoretically infinite, but practically moderate, multiple of n years.

ELASTICITIES OF EXPENDITURE IN THE DYNAMIC THEORY OF DEMAND

By Gerhard Tintner

The author studied in a previous paper[1] the influence of income, prices and interest rates upon the quantities of various commodities demanded at different points in time. It seems desirable to investigate also the influence of these factors upon the expenditures.[2] This can be done by a generalization and adaptation of the formulae already derived. This approach, by the way, is similar to the one used by Fisher[3] and recently by von Stackelberg.[4] It seems desirable to express the relationships in form of elasticities, since this makes them independent of the scale.

A. NOTATION

The individual is at the point of time 0 and plans for the points 1, 2, $\cdots$, n. There are m commodities. The utility of the individual depends upon all the quantities of all commodities which he expects to consume over the whole period. The individual maximizes this utility by adapting his consumption of various commodities at different points in time, with given expected incomes, interest rates, and prices for all commodities at all points in the whole interval 1, 2, $\cdots$, n.

Let x_{vs} be the quantity of the commodity v which he expects to consume at the point in time s and p_{vs} the price which he expects to pay for it. Let I_s be the expected income at the point s, i_s the expected rate of interest, and $r_s = 1+i_s$ the expected accumulation rate at the point s. We define $R_s = r_1 r_2 \cdots r_s$ as the total accumulation factor for the period 1, 2, $\cdots$, s, and denote by $J_s = I_s/R_s$ the expected discounted income at the point in time s and by $q_{vs} = p_{vs}/R_s$ the expected discounted price of the commodity v at the point in time s. The total discounted income $J = \sum_{s=1}^{n} J_s$. This is also the total expenditure since there is no net saving over the entire period 1, 2, $\cdots$, n.

Let $k_s = J_s/J$ be the proportion of the total discounted income expected at the point in time s. Define the discounted expected expenditure on commodity v at the point in time s as $e_{vs} = q_{vs}x_{vs}$. Then the

[1] G. Tintner, "The Theoretical Derivation of Dynamic Demand Curves," Econometrica, Vol. 6, 1938, pp. 375 ff. See also "The Maximization of Utility over Time," Econometrica, Vol. 6, 1938, pp. 154 ff.

[2] This idea was suggested by several members of the Chicago University Faculty Seminar at the occasion of a paper read on a related subject.

[3] I. Fisher, *The Theory of Interest*, New York, 1930, especially pp. 288 ff. Our analysis corresponds to Fisher's first approximation (pp. 99 ff.).

[4] H. von Stackelberg, "Beitrag zur Theorie des individuellen Sparens," *Zeitschrift für Nationalökonomie*, vol. 9, 1938, pp. 167 ff.

proportion of the total discounted expenditure (or total discounted income) spent on the commodity v at the point in time s is $K_{vs} = e_{vs}/J$. The total discounted expenditure on the commodity v at all points in time is $e_{v0} = \sum_{s=1}^{n} e_{vs}$. The ratio of the expenditure on the commodity v at the point in time s to the total expenditure on the commodity v is $K_{vs,v0} = e_{vs}/e_{v0}$. The total expenditure at the point s on all commodities is $e_{0s} = \sum_{v=1}^{m} e_{vs}$. The ratio of the expenditure on the commodity v at the point in time s to the total expenditure at the point s is $K_{vs.0s} = e_{vs}/e_{0s}$. The ratio of the expenditure on commodity v in the whole interval $1, 2, \cdots, n$, to the total income or expenditure is $e_{v0}/J = K_{v0}$ and the ratio of the expenditure on all commodities at the point s to the total expenditure is $e_{0s}/J = K_{0s}$.

The following relationships hold for the sums of these quantities:

$$\sum_{s=1}^{n} k_s = \sum_{s=1}^{n}\sum_{v=1}^{m} K_{vs} = \sum_{s=1}^{n} K_{vs,v0}$$

$$= \sum_{v=1}^{m} K_{vs,0s} = \sum_{s=1}^{n} K_{0s} = \sum_{v=1}^{m} K_{v0} = 1.$$

If $\sigma_{ut,vs}$ is the elasticity of substitution[5] of the commodity v at the point in time s for the commodity u at the point in time t we have the relationship: $\sum_{s=1}^{n}\sum_{v=1}^{m} K_{vs}\,\sigma_{ut,vs} = 0$. Denoting by Ex_{vs}/EJ the elasticity of the demand for the commodity v at the point s with respect to the total discounted income we also get: $\sum_{s=1}^{n}\sum_{v=1}^{m} K_{vs}\,(Ex_{vs}/EJ) = 1$.

B. ELASTICITIES OF EXPENDITURE ON A SPECIFIC COMMODITY AT A GIVEN POINT IN TIME

Let $e_{ut} = x_{ut}q_{ut}$ be the expected expenditure on the commodity u at the point in time t. Making use of the property that the elasticity of a constant multiple of a function equals the elasticity of this function[6] we get for the income elasticity of expenditure:

$$\frac{Ee_{ut}}{EJ} = \frac{Ex_{ut}}{EJ}\,. \tag{1}$$

If $w \neq t$, $z \neq u$, the price elasticity of expenditure is:

$$\frac{Ee_{ut}}{Eq_{zw}} = \frac{Ex_{ut}}{Eq_{zw}} = K_{zw}\left(\sigma_{ut,zw} - \frac{Ex_{ut}}{EJ}\right). \tag{2}$$

On the other hand, the price elasticity of expenditure with respect to

[5] J. R. Hicks, *Théorie mathématique de la valeur*, Paris, 1937, p. 14; *Value and Capital*, Oxford, 1939, pp. 227 ff.; R. G. D. Allen, *Mathematical Analysis for Economists*, London, 1938, pp. 512 ff.

[6] R. G. D. Allen, *op. cit.*, p. 252.

the expected price of the commodity in question (u) at the point in time in question (t) is:

$$\frac{Ee_{ut}}{Eq_{ut}} = \frac{Ex_{ut}}{Eq_{ut}} + 1 = K_{ut}\left(\sigma_{ut,ut} - \frac{Ex_{ut}}{EJ}\right) + 1. \tag{3}$$

The elasticities with respect to the accumulation rates are, if $w>t$:

$$\frac{Ee_{ut}}{Er_w} = \frac{Ex_{ut}}{Er_w} = \sum_{s=1}^{w-1}(k_s - K_{0s})\frac{Ex_{ut}}{EJ} + \sum_{s=1}^{w-1}\sum_{v=1}^{m} K_{vs}\sigma_{ut,vs}. \tag{4}$$

But if $w \leqq t$, the accumulation-rate elasticity becomes:

$$\frac{Ee_{ut}}{Er_w} = \frac{Ex_{ut}}{Er_w} - 1 = \sum_{s=1}^{w-1}(k_s - K_{0s})\frac{Ex_{ut}}{EJ} + \sum_{s=1}^{w-1}\sum_{v=1}^{m} K_{vs}\sigma_{ut,vs} - 1. \tag{5}$$

It should be noted that the quantity $\sum_{s=1}^{w-1}(k_s - K_{0s})$ is the ratio of the expected discounted saving in the period $1, 2, \cdots, w-1$ to the total expected income or expenditure in the period $1, 2, \cdots, n$. This saving may of course be positive or negative.

C. ELASTICITIES OF THE EXPENDITURE ON A GIVEN COMMODITY AT ALL POINTS IN TIME

We define $e_{u0} = \sum_{t=1}^{n} e_{ut}$ as the expenditure on the commodity u over the whole period $1, 2, \cdots, n$. It follows from a theorem about the elasticity of a sum of functions[7] that the elasticity of e_{u0} with respect to any quantity y is equal to:

$$\frac{Ee_{u0}}{Ey} = \sum_{t=1}^{n} K_{ut,u0}\frac{Ee_{ut}}{Ey}. \tag{5^1}$$

This theorem is very helpful in the derivation of the elasticities of the total expenditure on a given commodity.

We have for the income elasticity:

$$\frac{Ee_{u0}}{EJ} = \sum_{t=1}^{n} K_{ut,u0}\frac{Ex_{ut}}{EJ}; \tag{6}$$

and for the price elasticities, if $u \neq z$:

$$\frac{Ee_{u0}}{Eq_{zw}} = \sum_{t=1}^{n} K_{ut,u0}\frac{Ex_{ut}}{Eq_{zw}} = K_{zw}\sum_{t=1}^{n} K_{ut,u0}\left(\sigma_{ut,zw} - \frac{Ex_{ut}}{EJ}\right). \tag{7}$$

But the price elasticities for the price of the commodity in question (u) at the point in time w are:

[7] R. G. D. Allen, *op. cit.*, p. 252.

$$
(8)\qquad
\begin{aligned}
\frac{Ee_{u0}}{Eq_{uw}} &= \sum_{t=1}^{n} K_{ut,u0}\frac{Ex_{ut}}{Eq_{uw}} + K_{uw,u0} \\
&= K_{uw}\sum_{t=1}^{n} K_{ut,u0}\left(\sigma_{ut,uw} - \frac{Ex_{ut}}{EJ}\right) + K_{uw,u0}.
\end{aligned}
$$

Finally, the accumulation-rate elasticities are:

$$
(9)\qquad
\begin{aligned}
\frac{Ee_{u0}}{Er_{w}} &= \sum_{t=1}^{n} K_{ut,u0}\frac{Ex_{ut}}{Er_{w}} - \sum_{t=w}^{n} K_{ut,u0} \\
&= \sum_{s=1}^{w-1}(k_s - K_{0s})\sum_{t=1}^{n} K_{ut,u0}\frac{Ex_{ut}}{EJ} \\
&\quad + \sum_{t=1}^{n}\sum_{s=1}^{w-1}\sum_{v=1}^{m} K_{ut,u0}K_{vs}\sigma_{ut,vs} - \sum_{t=w}^{n} K_{ut,u0}.
\end{aligned}
$$

The last sum appearing in formula (9) is the ratio of the discounted expenditure on the commodity u in the period $w, w+1, \cdots, n$ to the total expenditure on this commodity over the entire period $1, 2, \cdots, n$.

D. ELASTICITIES OF THE EXPENDITURE ON ALL COMMODITIES AT A GIVEN POINT IN TIME

This part of our analysis comes nearest to the ideas first expressed by Irving Fisher in his celebrated theory of interest.[8] We define $e_{0t} = \sum_{u=1}^{m} e_{ut}$ the expenditure on all commodities at the point in time t. A well-known theorem on elasticities of sums of functions yields again a useful formula:

$$
(10)\qquad \frac{Ee_{0t}}{Ey} = \sum_{u=1}^{m} K_{ut,0t}\frac{Ee_{ut}}{Ey}.
$$

The elasticities of the total expenditure on all commodities at the point in time t with respect to the total discounted income are:

$$
(11)\qquad \frac{Ee_{0t}}{EJ} = \sum_{u=1}^{m} K_{ut,0t}\frac{Ex_{ut}}{EJ}.
$$

If $w \neq t$, the price elasticities of the total expenditure on all commodities at the point t with respect to the discounted price of the commodity z at the point in time w are:

$$
(12)\qquad \frac{Ee_{0t}}{Eq_{zw}} = \sum_{u=1}^{m} K_{ut,0t}\frac{Ex_{ut}}{Eq_{zw}} = K_{zw}\sum_{u=1}^{m} K_{ut,0t}\left(\sigma_{ut,zw} - \frac{Ex_{ut}}{EJ}\right).
$$

[8] I. Fisher, *op. cit.* See also *The Rate of Interest*, New York, 1907.

On the other hand, the elasticity of the total expenditure on all commodities at the point in time t with respect to the discounted price of the commodity z at the same point in time is:

$$
\begin{aligned}
\frac{Ee_{0t}}{Eq_{zt}} &= \sum_{u=1}^{m} K_{ut,0t}\frac{Ex_{ut}}{Eq_{zt}} + K_{zt,0t} \\
&= K_{zt}\sum_{u=1}^{m} K_{ut,0t}\left(\sigma_{ut,zt} - \frac{Ex_{ut}}{EJ}\right) + K_{zt,0t}.
\end{aligned} \tag{13}
$$

Finally the elasticities of expenditure on all commodities at a given point in time with respect to the accumulation rate are for $w>t$:

$$
\begin{aligned}
\frac{Ee_{0t}}{Er_w} &= \sum_{u=1}^{m} K_{ut,0t}\frac{Ex_{ut}}{Er_w} = \sum_{s=1}^{w-1}(k_s - K_{0s})\sum_{u=1}^{m} K_{ut,0t}\frac{Ex_{ut}}{EJ} \\
&\quad + \sum_{u=1}^{m}\sum_{s=1}^{w-1}\sum_{v=1}^{m} K_{vs}K_{ut,0t}\sigma_{ut,vs}.
\end{aligned} \tag{14}
$$

If $w \leqq t$, then the elasticity of expenditure with respect to the accumulation rate becomes:

$$
\begin{aligned}
\frac{Ee_{0t}}{Er_w} &= \sum_{u=1}^{m} K_{ut,0t}\frac{Ex_{ut}}{Er_w} - 1 = \sum_{s=1}^{w-1}(k_s - K_{0s})\sum_{u=1}^{m} K_{ut,0t}\frac{Ex_{ut}}{EJ} \\
&\quad + \sum_{s=1}^{w-1}\sum_{v=1}^{m}\sum_{u=1}^{m} K_{vs}K_{ut,0t}\sigma_{ut,vs} - 1.
\end{aligned} \tag{15}
$$

The last term is minus one since $\sum_{u=1}^{m} K_{ut,0t}=1$.

The difference between our approach and the theory of Fisher[9] is the following: We assume explicitly that the utility function or indifference map of the individual depends not merely upon the undiscounted expenditure stream $R_1e_{01} \cdots R_ne_{0n}$ (Fisher's income stream), but upon all the quantities of all commodities which the individual plans to consume at all points in time between 1 and n: $x_{11}, \cdots, x_{mn}$. The time preference, if any, should be expressed in the form of the utility function.

Iowa State College
Ames, Iowa

[9] I. Fisher, *op. cit.;* H. von Stackelberg, *loc. cit.*

THE DEMAND FOR PASSENGER CARS IN THE UNITED STATES: A REPLY

By Robert Solo

Dr. P. de Wolff's paper, "The Demand for Passenger Cars in the United States," appeared in the April, 1938 issue of Econometrica. In the beginning of this paper the author correlates the deviations from the trend of new-car registrations and scrappings with changes in the national income, as measured by changes in nonworkers' income and corporation profits. From the resultant regression equation, he attempts to obtain the elasticities of automobile demand for the years studied (1921–1934). It is with this latter attempt that I am concerned here.

Since they are the basis of his later calculations I shall briefly restate de Wolff's earlier equations. P represents the production of new cars, S the annual scrappings that actually occurred, and S_b the number of scrapped cars when the effects of the cycle are removed. Thus $P-S$ is the annual increase in the number of registered cars, J denotes the nonworkers' income during the year, J_{-1} nonworkers' income during the previous year, t the number of years passed since 1921. The following equation is derived:

$$100(S - S_b)/S_b = 2.24J + 1.00J_{-1} - 98.5 - 1.64t, \tag{1}$$

or

$$100(S - S_b)/D_b = 3.24(0.69J + 0.31J_{-1}) - 98.5 - 1.64t.$$

Denoting nonworkers' income by $J_{-0.31}$ the relationship is changed to

$$100(S - S_b)/S_b = 3.24[J_{-0.31} - (29.7 + 0.51t)]. \tag{2}$$

The following equation is derived as a result of the above calculations:

$$A = -0.65K + 0.20N + 3.36; \tag{3}$$

where A = deviation between $P-S$ and its trend, in millions of cars,
K = price of cars, in hundreds of dollars,
N = total corporation profits, in billions of dollars.

With the help of (3) de Wolff attempts to derive an equation that will tell him the elasticity of demand at different points of time:

$$\text{Elasticity} = \epsilon = \frac{\partial P}{\partial K} : \frac{P}{K}$$

$$= \frac{\partial A}{\partial K} : \frac{P}{K} = 0.65\frac{K}{P}\cdot$$

The results he gets are as follows:

Year	1921	1922	1923	1924	1925	1926	1927	1928	1929	1930	1931	1932	1933	1934
Elasticity of total demand for passenger cars	3.3	2.0	1.1	1.3	1.3	1.3	1.8	1.3	1.0	1.5	2.0	3.4	2.2	1.7

Here is an astonishing deduction, for de Wolff finds that in times of prosperity the elasticity of demand for automobiles, a durable luxury product, grows less and in times of depression elasticity of demand increases. Since there is no other definition given, I assume that the meaning of elasticity of demand is the generally accepted one, referring to a demand schedule at a point of time, in this case being the average approximation for the year. I need not elaborate on the effect that such a conclusion as de Wolff draws, if it is accepted, must have on economic thinking. The usual conclusions with regard to the changed shape of the demand curve of such products as automobiles is exactly the reverse of those which de Wolff postulates. It is difficult to see why a depression should induce greater elasticity into the demand curve for automobiles. The luxury buyers are almost entirely sloughed off. The new cars sold are the result of relative necessity. Under these conditions one would expect the new demand to be less, rather than more elastic.[1] Of course, the reverse may be true. It may be that car buyers, buying for practical utilitarian reasons, are more sensitive to price concessions, than those buyers to whom the car is a luxury product. The fact is that de Wolff's point is most important. If true, it would be exceedingly difficult to find any logic at all in the rigid prices of the automobile oligopoly (or monopoly). If, as the demand curve moves to the left, it at the same time becomes more elastic, it will always be to the best interest of the monopolist to lower his price in order to maximize his profit, or to minimize his loss. Perhaps, if de Wolff is right, the auto magnates do not know the course of their own, and incidentally the nation's, best interest. If this is true, it is a most important observation. But is it true? It may be, but I am unwilling to accept the proposition as it has been brought forward by de Wolff. I believe that the proof and method of de Wolff's can be challenged and found wanting at three points.

[1] "A fall in demand due to cyclical movements in trade is likely to be accompanied by a reduction in the elasticity in the case of durable goods, the replacement of which can be postponed to better times, for only the most urgent demand for goods will be effective during the slump even if prices are considerably reduced."—Joan Robinson, *Economics of Imperfect Competition*, p. 73.

It is true that the slope of the curve must increase as demand moves to the left, so that elasticity might decrease or even remain the same. Thus, if depression meant a mere sloughing off of buyers, the elasticities of the demands of the buyers remaining being unchanged, a decrease in demand would mean a higher elasticity.

I. Can the shapes or elasticities of different demand curves, existing at different points of time be derived from the shape of a historic demand curve connecting points on these individual demand curves? I think not, and find no substantial relationship between the elements from which the linear equation is derived and the demand curve for autos at any point of time. How is one to know that the changed relationships being measured are due to changes in elasticity or to changes in other factors, as the shifting of demand curves. After all, even if there were no change in the elasticity of the actual demand curve, but only a shifting of demand toward the right or left, which followed a fairly definite pattern during the up and down swings of the business cycle, one would get consistent results as to the change in elasticity following the formula, although no change actually occurred.

II. Aside from questioning its derivation, I find that the equation $\epsilon = 0.65PK/KP$ when examined by itself is defective as a measure of anything approaching the elasticity of demand for autos at a point of time. What the formula actually measures, or attempts to measure, is the relative shrinkage or expansion of cars produced in any one year, with relation to the changes in the average price of cars for that year. The meaning of high elasticity is that a small change (as a lowering in the price) will cause a great change (as a great increase) in the number of cars produced. This is relative, and I speak in relative terms. One therefore would expect that, in periods where there was a slight change in price and a great change in production, the demand curve would be given a high elasticity, and in instances where there was a great change in price and a small change in production the demand curve would be assigned a low elasticity. However, the nature of the formula is such that just the reverse must be true. The examples below are self-explanatory. In the first instance, the change of price is infinitesimal. In the second instance, the change in production is infinitesimal. In each instance the result gives a direction to the change in elasticity which is the reverse of the true direction of change, as the meaning of elasticity is generally understood.

1. (a) $0.65 \dfrac{600}{1000} = 3.9.$

   (b) $0.65 \dfrac{600}{2000} = 1.95.$

2. (a) $0.65 \dfrac{600}{1000} = 3.9.$

   (b) $0.65 \dfrac{800}{1000} = 5.2.$

III. Aside from the defects in the derivation of the formula, and in the formula itself, the method of calculating the data used in the formula is clearly defective. I speak with reference to the method used to calculate the price of cars at any point of time (dividing the total number of cars produced into the total expenditure). This method considers the upward shifting in the average price of the car, due to shifting in the consumer demand for cars of better grade in time of prosperity and the reverse shifting during period of depression, as real changes in the price demanded for cars.

No attempt has been made to find the exact extent to which automobile price indexes are influenced by this shifting of preferences toward lower-price cars during depression and toward higher-price cars during prosperity, or by the increase in number of expensive models

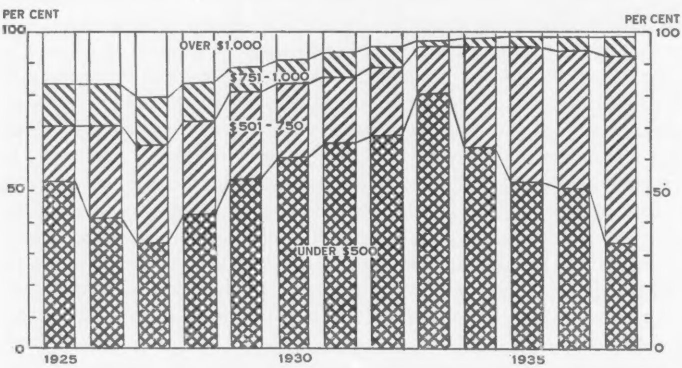

Figure 1.—Factory sales of motor cars in United States and Canada by wholesale price classes. Source: Automobile Manufacturers Association, *Facts and Figures, 1938*, p. 10.

that standard-brand producers put on the market during prosperity and rising income, and by the increases of cheaper models that these producers put on the market when the national income is decreasing. It is not, however, difficult to show that these shifts are of considerable importance. I have listed a few of the ways that the significance of preference shifts and model changes can be indicated.

1. In examining Figure 1 (from *Facts and Figures, 1938,* published by the Automobile Manufacturers Association) it can be seen that there is an inverse relationship between increases and decreases in the national income and the percentage of cars sold whose price is under $500 wholesale. In 1932, 81 per cent of the cars sold had a wholesale

price of under $500, while in 1937, only 33 per cent of the cars sold were under $500 wholesale. During this time period the standard models of Ford, Plymouth, and Chevrolet, forming over 95 per cent of the cars sold in this price class, which had been under the $500 wholesale mark, remained under the $500 wholesale price mark.

2. In Table 1 a tabulation is made of the increases and decreases

TABLE 1

From 1936 to 1937 there was an increase of approximately 12.9 in terms of national income (computed from the rise of an average of 12.9 in the first six months in the index of National Income given by the Bureau of Agricultural Economics). The index change was from 86.8 to 99.7.

Price of 1937 2-door sedan	Number of makes showing change in share of market from 1936 to 1937	
	Decrease	Increase
$1200 and above	2	2
$850–$1199	1	5
$735–$ 849	0	5
$667–$ 734	2	1
Below $667	3*	0

* The "Big Three."

in the percentage of the market that went to cars, placed within broad price ranges, during a year of rising income. The trend away from cheap to high- and medium-price brands is evident.

3. Figure 2, prepared by the author, shows the following:

(a) Simple price of a single standard Ford model in order to show the extent of actual fluctuation that took place in the price demanded by the manufacturer.

(b) The arithmetic average of all models of Ford, in order to show the effect of the producer's changing models.

(c) The yearly average price of three standard models of Ford, weighted by the number of models sold for the year, in order to show the effect of changed consumer preferences among these three models.

It is seen that (a) remains comparatively stable while (b) and (c) fluctuate violently with changes in business conditions.

This means of calculating the price gives an index that is not compatable with a study of this sort. The object of the formula is to find the effects of real changes in price on the quantity of cars demanded. This method of calculating price will not only produce an inevitable distortion in the results, but a distortion which will regularly tend to reverse the true trend. A small number of cars bought at a high price

does not necessarily mean that the high price affected consumption. It may signify a changed nature of demand for autos.

Increased expenditure may have gone into the purchasing of better rather than more cars, and there may have been no change in the price that manufacturers asked for cars of the same quality. The only

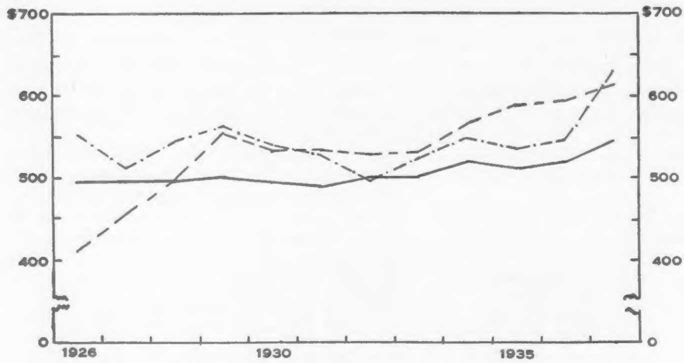

FIGURE 2.—Prices of Ford motor cars.
———— Simple price of 2-door sedan.
— — — Arithmetic average price of all models.
— · — · — Average price of three standard models, weighted by annual sale of those models.

way to rule out this distortion would be to use a price index computed from the real price changes that actually occurred in standard car models and to equate these changes against the changes in total expenditures, not against changes in total car production. Even here it would be difficult to take the introduction of new models into account.

Washington, D. C.

THE DEMAND FOR PASSENGER CARS IN THE UNITED STATES: A REJOINDER

By P. de Wolff

In a paper, published in this journal some time ago (Vol. 6, April, 1938, pp. 113–129), I showed the possibility of giving a satisfactory explanation of the demand for passenger cars in the United States by the following simple scheme. Total demand for cars is divided into two components: the demand for replacement and the demand for first purchase. The first component is explained with the aid of a normal lifetime distribution of cars, disturbed by cyclical influences. The trend of the second component shows clearly the character of a saturation curve, whereas the deviations from the trend (A) are explained by cyclical changes of corporation profits (N) and variations of the average car price (K). Between A, K, and N a relation was found of the following form:

$$A = -0.65K + 0.20N + 3.36. \tag{1}$$

(A in millions of cars; K in hundreds of dollars; N in billions of dollars.) In the preceding paper Mr. Solo has criticized this formula and some of the conclusions drawn from it. Mr. Solo's criticisms are summarized in three paragraphs which I will meet in turn.

In paragraph I, page 273, Mr. Solo is extremely sceptical about the possibility of deriving "the shapes or elasticities of different demand curves, existing at different points of time, from the shape of a historic demand curve, connecting points on these individual demand curves." The method I used is an application of the Walras scheme of demand and supply. Although this scheme already has been applied often in statistical investigations (cf. the works of Schultz, Frisch, Tinbergen, and others) its possibilities for deriving demand functions from empirical data seem to be rather unknown to many economists. The fundamental assumption of the Walras method is that the quantity D of a certain good, demanded per unit of time, depends not only on the price K but also on a number of other factors, called demand factors, $d_1, \cdots, d_m$. These factors may be different for different goods; the most plausible factors are income and prices of competing goods. The relation between D, K, and $d_1, \cdots, d_m$ can be represented in the following way:

$$D = D(K, d_1, \cdots, d_m).$$

At the same time it is assumed that supply S depends on price and certain supply factors $s_1, \cdots, s_n$. It can in turn be represented by an analogous expression

$$S = S(K, s_1, \cdots, s_n).$$

Further it is assumed that the numbers of buyers and sellers are sufficiently large so that no one individually can exert an influence on price and consequently every one considers price as a given quantity. During a certain unit of time, with fixed values of $d_1, \cdots, d_m$ and $s_1, \cdots, s_n$, the quantity actually sold P and the price K will then be determined by the equations $D=S=P$, expressing that, at every moment, the quantities demanded and supplied must be equal to the quantity sold. Now suppose that the factors $d_1, \cdots, d_m$ which enter into the demand function are known; then it is possible to determine approximately the function D. In fact, for every unit of time (e.g., years) we know the values of P, K, $d_1, \cdots, d_m$ and the numbers form a solution of the equation $D=P$. Thus, of this equation we know as many solutions as the number of observations. It is a problem of mathematical statistics to determine how much knowledge of the function D can be derived from the observed data and this need not be discussed here. It was only my purpose to show the principal possibility of obtaining D. A very instructive treatment of the topic is to be found in Schultz's last book.[1]

It is extremely important to observe that the function D can be determined without making any assumption about the character of S, for, although changes in S will exert an influence on the values of P and K, they do not change the fact that every set of observed values of P, K, $d_1, \cdots, d_m$ must form a solution of the equation $D=P$. Therefore the method can also be applied to determine D, if the supply side of the market is ruled by imperfect competition. The only provision is that the shape of D does not change. It makes no difference to the consumers, whether the price, which they regard as a given quantity, as a matter of fact has been fixed by their interaction with a great number of suppliers or whether it has been fixed by one monopolist. The difference is on the supply side. For example, in the case of monopoly the equation $P=S$ no longer plays a role in fixing K, but it must be interpreted as determining the quantity S, which has to be supplied in order to adapt supply to demand. From the foregoing it will be clear that, accepting Walras' scheme, the difficulty of determining demand functions consists wholly in finding the factors $d_1, \cdots, d_m$. It is evident that this problem can be discussed only for every individual commodity. In my case it proved to be sufficient to include in the demand function, besides price, only one variable, namely income N, but it cannot be

[1] H. Schultz, *The Theory and Measurement of Demand*, University of Chicago Press, 1938.

denied that a more refined analysis of the problem would make it necessary to take other factors into consideration.

Mr. Solo's second argument concerns the formula for the elasticity of demand ϵ which can be derived from (1). As explained at the beginning of this paper the whole demand P is split up in several parts of which only A is dependent on price. The ordinary formula for the elasticity becomes therefore:

$$\epsilon = -\frac{\partial P}{\partial K}\cdot\frac{K}{P} = -\frac{\partial A}{\partial K}\cdot\frac{K}{P} = 0.65\,\frac{K}{P}\,. \tag{2}$$

I note explicitly that this is the "generally accepted one." However, Mr. Solo seems to have entirely misunderstood the meaning of this simple expression. He is confusing at least two things. The first one is that in order to calculate elasticity, one has to compare two states, *differing only in the values of price and quantity demanded;* and the second one is that the actual value of ϵ will show fluctuations in time. This can most easily be demonstrated by Mr. Solo's "self-explanatory examples" on page 273. In the first of these examples he compares two situations, in both cases $K=6$, but $P=1$ in the first and $P=2$ in the second case (in my units!). It is obvious that my formula will yield two values for ϵ for these two situations of which the first one must be twice as great as the second one. But it is meaningless to use these two situations, as Mr. Solo suggests, for a calculation of the price elasticity connected with a transition of the first to the second situation. For K having remained constant, the change in P can only have been caused by changes in N and therefore it is obvious that the conditions for the calculation of ϵ are not fulfilled. Moreover, it is not to be seen which of the two values of ϵ Mr. Solo will assign to the transition. (Of course each of these values has its meaning: it measures the elasticity of changes in demand, caused by changes in price, in each of the two situations considered.) Analogous objections can be made against his second example.

Mr. Solo is convinced that the results obtained with my formula are wrong. I used the formula to compute the value of ϵ for the years 1921–1934 and found (a) $\epsilon>1$ and (b) ϵ moving antiparallel to the business-cycle pattern.

In regard to (a) Mr. Solo writes: "If, as the demand curve moves to the left, it at the same time becomes more elastic, it will always be to the best interest of the monopolist to lower his price in order to maximize his profit, or to minimize his loss." So formulated the argument cannot be right. Whether gross income will increase or de-

crease with price depends on whether $\epsilon>1$ or $\epsilon<1$. (I suppose that Mr. Solo means gross income instead of profit. The relation between profit and price can only be analyzed if it is known how costs depend on output and this aspect of the problem has been left entirely out of consideration.) But as I found that $\epsilon>1$ I am convinced that the gross income of the motor-car industry will increase as prices are lowered. It is, however, a strange procedure to deduce the incorrectness of my results from the price policy of the car magnates which may be governed by many other factors, even if they are quite aware of their own interest. Moreover, I cannot see that, supposing my result to be right, it forms such an "important observation." I thought that it was rather generally accepted that the demand for luxury products, like motor cars, would be elastic.

In regard to (b) Mr. Solo believes that in reality ϵ will move parallel to the business cycle. His opinion is based on the fact that during a depression new cars will be bought only in cases of relative necessity. This, however, is not very convincing. In my analysis demand consists of two parts of which the demand for replacement is totally independent of price. Still I find that ϵ is decreasing when prosperity is increasing. In order to obtain a more satisfying answer to our question it will be necessary to study the behavior of ϵ more in detail. It is easy to see that this behavior will be determined by the sign of $\partial\epsilon/\partial N$.

Now
$$\epsilon = -\frac{\partial P}{\partial K}\cdot\frac{K}{P} = -K\frac{\partial \log P}{\partial K},$$

and therefore

$$\frac{\partial \epsilon}{\partial N} = -K\frac{\partial^2 \log P}{\partial K \partial N}\cdot$$

(Since the difference between A and P is dependent on income it is now not allowable to replace P by A, as in the formula for ϵ itself, and it must be remembered that the coefficient P_{01} which will be used in the following is not equal to 0.20 as in formula (1) but has a greater value.[2])

We shall now study the behavior of the demand function in the neighbourhood of a point, characterized by the values $\overline{K}$ and $\overline{N}$ of K and N. We denote the deviations of K and N from $\overline{K}$ and $\overline{N}$ by k and n

[2] Strictly speaking a variable I different from N has been used in the explanation of $P-A$, but on the whole these two variables I and N show practically the same movement and their difference will be neglected in this connection.

and then it is always possible to expand P in a power series with arguments k and n.

We put

$$P = p_{00} + p_{10}k + p_{01}n + p_{20}k^2 + p_{11}kn + p_{02}n^2 + \cdots$$
$$= p_{00}[1 + p_{10}'k + p_{01}'n + p_{20}'k^2 + p_{11}'kn + p_{02}'n^2 + \cdots].$$

Then, by putting $\quad p_{2k}' = \dfrac{p_{2k}}{p_{00}},\quad$ we have:

$$P = \log p_{00} + p_{10}'k + p_{01}'n + p_{20}'k^2 + p_{11}'kn + p_{02}'n^2 + \cdots$$
$$- \tfrac{1}{2}(p_{10}'k + p_{01}'n + \cdots)^2 + \cdots$$
$$= \log p_{00} + p_{10}'k + p_{01}'n + (p_{20}' - \tfrac{1}{2}p_{10}'^2)k^2$$
$$+ (p_{11}' - p_{10}'p_{01}')kn + (p_{02}' - \tfrac{1}{2}p_{01}'^2)n^2 + \cdots .$$

From this expression it follows immediately that:

$$\frac{\partial\epsilon}{\partial N} = -\overline{K}(p_{11}' - p_{10}'p_{01}') = \frac{\overline{K}}{p_{00}{}^2}(p_{10}p_{01} - p_{00}p_{11}).$$

Now it is evident that $p_{10}<0$ and $p_{01}>0$ and therefore the sign of $\partial\epsilon/\partial N$ depends wholly on the sign and magnitude of p_{11}. If $p_{11}>0$, then $\partial\epsilon/\partial N$ always <0. The same holds true, when $0>p_{11}>p_{10}p_{01}/p_{00}$. But if $p_{11}<p_{10}p_{01}/p_{00}$, then $\partial\epsilon/\partial N>0$. Now in my formula $p_{11}=0$ and therefore $\partial\epsilon/\partial N$ must be negative. This, however, proves nothing; the quadratic terms have been left out, not because the coefficients p_{20} p_{11}, and p_{02} are small compared with p_{10} and p_{01}, but because the products $p_{20}k^2$, $p_{11}kn$, and $p_{02}n^2$ are small compared with $p_{10}k$ and $p_{01}n$. On these grounds I must admit that the systematic changes of ϵ which I found do not deserve much credit. In this connection it is of interest to note that it follows from the foregoing that a formula of the type (1) may be used to compute satisfactory values for the changes in demand and even for the average value of ϵ (e.g., it seems to be statistically proved that the average elasticity during the examined period has been >1), *whereas it can not be used to determine, with the same degree of elasticity, the variations of* ϵ.

From the preceding argument it follows that the sign of $\partial\epsilon/\partial N$ cannot be determined without estimating the neglected coefficient p_{11}. Calculations made for this purpose showed that, even when a fairly wide margin of error was assigned to the values of p_{11}, the expression $(p_{10}p_{01}-p_{00}p_{11})$ proved to be negative. Therefore, granting that it is not possible to estimate the exact variation of ϵ from (1), I cannot

believe that Mr. Solo is right in assuming that ϵ will move parallel to the business-cycle pattern.

Mr. Solo's third argument is the most important one. The price index for motor cars, used in my study, has been obtained by dividing the number of cars produced into the value of total sales. Mr. Solo is quite right in stating that in this way shifts in consumer preferences may be recorded as changes in price. And it is quite clear that, if the real course of prices should be entirely different from the series I used, formula (1) and the conclusions based upon it would lose their value. It is, however, difficult to estimate the importance of consumers' shifts from the material which Mr. Solo puts forward. The shifting, represented by Figure 1 will, at least in part, have been caused by changes in prices. His different price indices for Ford cars are very interesting, but they do not permit drawing a conclusion for all cars.

The index which I used has been chosen on two grounds as I explained in my original paper. It shows nearly exactly the same fluctuations as the index of car prices compiled by the U. S. Bureau of Labor Statistics and it is recommended by Scoville, statistician of the Chrysler Corporation. I admit, however, that Mr. Solo is principally right in his last point and I should be very grateful to be supplied with a more satisfactory price index which would enable me to correct my results in this respect.

The Hague, Netherlands

COMMITTEE ON COST-PRICE RELATIONSHIPS

THE CONFERENCE ON PRICE RESEARCH has created a new standing committee on cost-price relationships. The principal objectives of this committee are: (1) formulation of research problems in the field; (2) clarification of concepts and terminology; (3) improvement of data; (4) improvement of methods of research; (5) initiation and stimulation of research.

In the preparation of a unified program of important and practicable price-cost research projects, the committee is examining critically a comprehensive list of problems in order to appraise the significance of each for economic analysis and for business administration and to determine its feasibility for immediate investigation.

The committee hopes also to be of assistance in the clarification of concepts and standardization of terminology so urgently needed if integration of economic theory, accounting procedure, and administrative practice is to be achieved. Lack of agreement on the meaning of terms lies at the root of much confusion in the discussion of cost-price relationships and impedes co-operation among accountants, economists, and business executives.

In working toward improvement of data this group plans to survey published cost material, appraise the adequacy of the information provided by cost accounts for the study of cost-price relationships, and use its influence to make the cost data of private enterprise available to investigators. The committee, therefore, hopes to examine the data with a view to framing cost-price research projects that are adapted to the available information and to improving and standardizing accounting data in order to make it more suitable for answering the questions economists and executives ask.

In attempting to improve methodology, the committee has begun an inventory of published research and has commissioned its staff assistant to prepare a monograph on methods of analyzing the behavior of the short-run costs of individual firms.

The committee hopes to stimulate and co-ordinate research in cost-price relationships as well as to initiate and supervise certain projects of its own. With this in view, it is hoped that persons interested and active in cost-price research will help this group keep in touch with research currently in progress.

The members of the committee are: Professor Edward Mason, Chairman, Harvard University; Mr. C. M. Armstrong; Professor J. M. Clark, Columbia University; Dr. Joel Dean, National Bureau of Economic Research; Mr. C. Oliver Wellington, McKinsey, Wellington

& Co.; Dr. R. H. Whitman, R. H. Macy & Co., Inc.; Dr. Theodore O. Yntema, University of Chicago.

Communications should be addressed to Joel Dean at the National Bureau of Economic Research, 1819 Broadway, New York City.

ELECTION OF FELLOWS

The Fellows of the Econometric Society have elected the four new Fellows, whose names and partial bibliographies follow:

OSKAR LANGE, University of Chicago, Chicago, Illinois.

Professor Lange was born in Poland. He has taught at the University of California, and is now Associate Professor of Economics at the University of Chicago.

Die Preisdispersion als Mittel zur statistischen Messung wirtschaftlicher Gleichgewichtsstörungen, Veröffentlichungen des Frankfurter Gesellschafts für Konjunkturforschung, 1932, 56 pp.

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WASSILY LEONTIEF, Harvard University, Cambridge, Massachusetts.

Professor Leontief received his Ph.D. at the University of Berlin in 1928. He is Assistant Professor of Economics at Harvard University.

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JOSIAH CHARLES STAMP, FIRST BARON SHORTLANDS, Park Hill Road, Shortlands, Kent, England.

Lord Stamp was educated at London University. He has held various official and business positions and since 1925 has been Chairman of the Board of the London, Midland, and Scottish Railway Co.

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THEODORE OTTE YNTEMA, School of Business, University of Chicago, Chicago, Illinois.

Professor Yntema received his A.B. from Hope College in 1921, A.M. from University of Illinois in 1922, and Ph.D. from University of Chicago in 1929. He is Professor of Statistics at the University of Chicago.

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FELLOWS OF THE ECONOMETRIC SOCIETY
JULY, 1939

Mr. R. G. D. ALLEN, London, England.
Professor LUIGI AMOROSO, Rome, Italy.
Professor OSKAR N. ANDERSON, Sofia, Bulgaria.
Dr. ALBERT AUPETIT, Paris, France.
Professor P. BONINSEGNI, Lausanne, Switzerland.
Professor ARTHUR L. BOWLEY, London, England.
Professor COSTANTINO BRESCIANI-TURRONI, Giza, Egypt.
*Professor CLEMENT COLSON, Paris, France.
Mr. ALFRED COWLES 3RD, Colorado Springs, Colorado, U.S.A.
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Professor FRANÇOIS DIVISIA, Paris, France.
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Professor GOTTFRIED HABERLER, Cambridge, Mass., U.S.A.
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Professor EDWIN B. WILSON, Boston, Mass., U.S.A.
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*Professor WLADYSLAW ZAWADZKI, Warsaw, Poland.
Professor F. ZEUTHEN, Copenhagen, Denmark.

* Deceased.

SUGGESTIONS FOR FELLOWSHIPS

The constitution of the Econometric Society states:

> All Fellows of the Society shall be nominated by the Council and elected by mail-vote of the Fellows. Such nomination may be made at any time. To be eligible for such nomination a person must have published original contributions to economic theory, or to such statistical, mathematical, or accounting analyses as have a definite bearing on problems in economic theory, and must have been a member of the Society for at least one year. Each year the Council shall offer the members an opportunity to suggest nominees for fellowships.

In accordance with this provision all members of the Econometric

Society are hereby invited to suggest nominees for fellowship. Each nominee should possess the following qualifications laid down by the Council:

1. He should be an economist.
2. He should be a statistician.
3. He should have some knowledge of higher mathematics.
4. He should have made original contributions to economic theory.

The present Fellows are listed on page 286, of this issue. The membership list published in ECONOMETRICA, Vol. 6, October, 1938, pp. 385–403, may be found convenient in determining those who are eligible for nomination in the present election.

Suggestions for fellowship, accompanied by biographical data and bibliographies of candidates, should be sent to Alfred Cowles 3rd, Secretary of the Econometric Society, 301 Mining Exchange Building, Colorado Springs, Colorado, U. S. A.

MEMBERSHIP LIST CHANGES

A new directory of members will be published in the October issue of ECONOMETRICA. Any errors in the October, 1938 list, or changes in address, should be reported at once to Alfred Cowles 3rd, Secretary of the Econometric Society, 301 Mining Exchange Building, Colorado Springs, Colorado, U. S. A.

CHANGE OF ADDRESS

The office of the secretary and treasurer of the Society and of the managing editor of ECONOMETRICA will be moved in September, 1939 from Colorado Springs to the University of Chicago. Mail to arrive after September 15 should be addressed as follows:

Alfred Cowles, Secretary and Treasurer
The Econometric Society
Social Science Building
University of Chicago
Chicago, Illinois, U. S. A.

Dickson H. Leavens
Managing Editor of Econometrica
Social Science Building
University of Chicago
Chicago, Illinois, U. S. A.